UNIVERSIDADE FEDERAL DO RIO DE JANEIRO

INSTITUTO ALBERTO LUIZ COIMBRA DE PÓS-GRADUAÇÃO E
PESQUISA DE ENGENHARIA – COPPE/UFRJ
PROGRAMA DE ENGENHARIA QUÍMICA

CURSO DE NIVELAMENTO PARA O MESTRADO
EQUAÇÕES DIFERENCIAIS

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1 - Solução da EDO de 1^a ordem

Spiegel, p. 162.

EQUAÇÃO DIFERENCIAL	**SOLUÇÃO**
Separação de Variáveis $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$	$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy = c$
Equação Linear de 1^a Ordem $\frac{dy}{dx} + P(x)y = Q(x)$	$ye^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$
Equação de Bernoulli $\frac{dy}{dx} + P(x)y = Q(x)y^n$	$\nu\,e^{(1-n)\int P\,dx} = (1-n)\int Q\,e^{(1-n)\int P\,dx}\,dx + c$ onde $\nu = y^{1-n}$. Se $n = 1$, a solução é $\ln\lvert y\rvert = \int (Q - P)dx + c$
Equação Exata $M(x,y)dx + N(x,y)\,dy = 0$ onde $\partial M/\partial y = \partial N/\partial x$	$\int M(x,y)dx + \int \lfloor N(x,y) - \int (\partial M/\partial y)dx \rfloor dy = c$
Equação Homogênea $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	$\ln\lvert x\rvert = \int \frac{d\nu}{F(\nu)-\nu} + c$ onde $\nu = y/x$. Se $F(\nu) = \nu$, a solução é $y = cx$.
$yF(xy)\,dx + xG(xy)dy = 0$	$\ln\lvert x\rvert = \int \frac{G(\nu)d\nu}{\nu\lvert G(\nu) - F(\nu)\rvert} + c$, onde $\nu = xy$. Se $G(\nu) = F(\nu)$, a solução é $xy = c$.

Série de Exercícios n^o 1 – EDO de 1^a ordem

***Referência**: F. Ayres Jr., "Theory and Problems of Differential Equations", Schaum Publishing, Nova Iorque (1952).*

(1) $x\dfrac{dy}{dx} = y + x^3 + 3x^2 - 2x$

R.: $2y = x^3 + 6x^2 - 4x\ln(x) + Cx$ (prob. 2, p. 35)

(2) $(1 + x^3)dy - (x^2 y)dx = 0$, $y(1) = 2$

R.: $y^3 = 4(1 + x^3)$ (prob. 5, p. 17)

(3) $(\cos(y) + y\cos(x))dx + (\operatorname{sen}(x) - x\operatorname{sen}(y))dy = 0$

R.: $x\cos(y) + y\operatorname{sen}(x) = C$ (prob. 1c, p. 26)

(4) $4x\,dy - y\,dx = x^2\,dy$

R.: $(x - 4)y^4 = Cx$ (prob. 3, p. 17)

(5) $(2x + 3y)dx + (y - x)dy = 0$

R.: $\ln(y^2 + 2xy + 2x^2) - 4\arctan\left(\dfrac{x + y}{x}\right) = C$ (prob. 10, p.18)

(6) $\begin{cases} xy' = 2y + x^3 e^x \\ y(1) = 0 \end{cases}$

R.: $y = x^2(e^x - e)$

2 - Solução da EDO linear de 2ª ordem

Forma geral:

$$\boxed{y''+P(x)y'+Q(x)y=R(x)}$$

Solução geral: $\boxed{y(x)=y_H(x)+y_P(x)}$

Solução da homogênea ($R(x)=0$): $y_H(x)=c_1y_1(x)+c_2y_2(x)$

Para obter $y_2(x)$ conhecendo $y_1(x)$: $y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}\,dx$

Solução particular: $y_P(x)=y_1(x)\int^x \frac{y_1(t)y_2(x)-y_1(x)y_2(t)}{W[y_1(t),y_2(t)]}\,R(t)\,dt$,

em que $W[y_1(t),y_2(t)]$ é o wronskiano, definido por

$$\boxed{W[y_1(t),y_2(t)]=\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}$$

Série de Exercícios n^o 2 – Equações Diferenciais Ordinárias Lineares de 2^a Ordem

(1) Determinar a solução geral de

$$xy''+2y'=\frac{1}{x}$$

sabendo que uma solução da homogênea é $y_1(x)=1$.

R.: $y(x)=c_1+\dfrac{c_2}{x}+\ln(x)$

(2) Determinar a solução geral da equação

$$xy''-(2x-1)y'+(x-1)y=0$$

sabendo que uma solução é $y_1(x)=e^x$.

R.: $y(x)=c_1e^x+c_2e^x\ln|x|$

(3) Resolver

$$\begin{cases} xy''-3y'+\dfrac{3}{x}y=x+2 \\ y_1(x)=x \end{cases}$$

R.: $y(x)=c_1x+c_2x^3-x^2-x\ln(x)$

(4) Resolver

$$\begin{cases} y''-\dfrac{2x}{1+x^2}y'+\dfrac{2}{1+x^2}y=0 \\ y_1(x)=x \end{cases}$$

R.: $y(x)=c_1x+c_2\left(x^2-1\right)$

Soluções da Equação Diferencial e de Diferenças com Coeficientes Constantes

Eq. Diferencial Ordinária	**Eq. de Diferenças**
$y''+a_1y'+a_2y=R(x)$	$y_{n+2}+a_1y_{n+1}+a_2y_n=R(n)$
$Dy=\dfrac{dy}{dx}\,,\ D^2y=\dfrac{d^2y}{dx^2}$ $D^{-1}f(x)=\int^x f(\eta)\,d\eta$ $e^D=E$	$Ey_n=y_{n+1}\,,\ E^2y_n=y_{n+2}$ $\Delta y_n=y_{n+1}-y_n$ $\Delta^2y_n=y_{n+2}-2y_{n+1}+y_n$ $E=\Delta+1$
$(D-r_1)(D-r_2)y=R(x)$	$(E-r_1)(E-r_2)y=R(n)$
$P(D)e^{rx}=e^{rx}P(r)$ $P(D)e^{rx}u(x)=e^{rx}P(D+r)u(x)$	$P(E)r^n=r^nP(r)$ $P(E)r^nu(n)=r^nP(rE)u(n)$
$\left\{\begin{matrix}\text{solução}\\ \text{geral}\end{matrix}\right\}=\left\{\begin{matrix}\text{solução da}\\ \text{homogênea}\end{matrix}\right\}+\left\{\begin{matrix}\text{solução}\\ \text{particular}\end{matrix}\right\}$	
$\begin{cases}y_H(x)=c_1e^{r_1x}+c_2e^{r_2x}\ ,\ r_1\neq r_2\\ y_H(x)=(c_1+c_2x)e^{r_1x}\ ,\ r_1=r_2\end{cases}$ $y_P(x)=e^{r_2x}\int^x e^{(r_1-r_2)\eta}\left[\int^\eta e^{-r_1t}\,R(t)\,dt\right]d\eta$	$\begin{cases}y_H(n)=c_1r_1^{\ n}+c_2r_2^{\ n}\ ,\ r_1\neq r_2\\ y_H(n)=(c_1+c_2n)r_1^{\ n}\ ,\ r_1=r_2\end{cases}$ $y_n(n)=\dfrac{1}{(E-r_1)(E-r_2)}R(n)$

$P(D)$ é um polinômio no operador D; $P(E)$ é um polinômio no operador E;

Solução da EDO de 2^a ordem com coeficientes constantes (Spiegel, p. 163)

Equação linear homogênea de 2^a ordem	Sejam m_1, m_2 as raízes de $m_2 + am + b = 0$. Então existem 3 casos. **Caso 1.** m_1, m_2 reais e distintas. $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ **Caso 2.** m_1, m_2 reais e iguais. $y = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$ **Caso 3.** $m_1 = p + qi$, $m_2 = p - qi$: $y = e^{px}\left(c_1 \cos qx + c_2 \operatorname{sen} qx\right)$ onde $p = -a/2$, $q = \sqrt{b - a^2/4}$.
$\dfrac{d^2 y}{dx^2} + a\dfrac{dy}{dx} + by = 0$ a, b são constantes reais	
Equação linear não-homogênea de 2^a ordem	Existem 3 casos correspondentes àqueles acima. **Caso 1.** $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dfrac{e^{m_1 x}}{m_1 - m_2}\displaystyle\int e^{-m_1 x} R(x)dx$ $+ \dfrac{e^{m_2 x}}{m_2 - m_1}\displaystyle\int e^{-m_2 x} R(x)dx$ **Caso 2.** $y = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + x e^{m_1 x}\displaystyle\int e^{-m_1 x} R(x)dx$ $- e^{m_1 x}\displaystyle\int x e^{-m_2 x} R(x)dx$ **Caso 3.** $y = e^{px}\left(c_1 \cos qx + c_2 \operatorname{sen} qx\right) + \dfrac{e^{px} \operatorname{sen} qx}{q}\displaystyle\int e^{-px}\ R(x) \cos qx\ dx$ $- \dfrac{e^{px} \cos qx}{q}\displaystyle\int e^{-px}\ R(x) \operatorname{sen} qx\ dx$
$\dfrac{d^2 y}{dx^2} + a\dfrac{dy}{dx} + by = R(x)$ a, b são constantes reais	
Equação de Euler ou Cauchy	Fazendo-se $x = e^t$, a equação se torna $\dfrac{d^2 y}{dt^2} + (a - 1)\dfrac{dy}{dt} + by = S\left(e^t\right)$ e pode ser resolvida como nos dois casos acima.
$x^2 \dfrac{d^2 y}{dx^2} + ax\dfrac{dy}{dx} + by = S(x)$	

Série de Exercícios n^o 3 - Equação Diferencial Ordinária com Coeficientes Constantes e Equação de Diferenças com Coeficientes Constantes

(1) $y''+y'-2y=0$, $y(0)=1$ e $y'(0)=1$
R.: $y(x)=e^{x}$

(2) $y''+4y'+5y=0$, $y(0)=1$ e $y'(0)=0$
R.: $y(x)=e^{-2x}\cos(x)+2e^{-2x}\operatorname{sen}(x)$

(3) $y''+2y'=3+4\operatorname{sen}(2x)$
R.: $y(x)=c_1+c_2e^{-2x}+\frac{3}{2}x-\frac{1}{2}\operatorname{sen}(2x)-\frac{1}{2}\cos(2x)$

(4) $y''+4y=x^2+3e^{x}$, $y(0)=0$, $y'(0)=2$
R.: $y(x)=\frac{7}{10}\operatorname{sen}(2x)-\frac{19}{40}\cos(2x)+\frac{1}{4}x^2-\frac{1}{8}+\frac{3}{5}e^{x}$

(5) $y''+5y'+4y=3-2x$
R.: $y(x)=c_1e^{-x}+c_2e^{-4x}-\frac{1}{2}x+\frac{11}{8}$

(6) $y''+9y=x\cos(x)$
R.: $y(x)=c_1\cos(3x)+c_2\operatorname{sen}(3x)+\frac{1}{8}x\cos(x)+\frac{1}{32}\operatorname{sen}(x)$

(7) $y''+{\omega_0}^2y=\cos(\omega t)$, $\omega\neq\omega_0$
R.: $y(x)=c_1\cos(\omega_0 t)+c_2\operatorname{sen}(\omega_0 t)+\frac{1}{{\omega_0}^2-\omega^2}\cos(\omega t)$

(8) $$\begin{cases} \dfrac{d^2x_1}{dt^2} - 4\dfrac{dx_1}{dt} + 4x_1 + 3\dfrac{dx_2}{dt} = 1 \\ \dfrac{dx_1}{dt} - 2x_1 + \dfrac{dx_2}{dt} + 2x_2 = 0 \\ x_1(0) = 1, \quad \left(\dfrac{dx_1}{dt}\right)_{t=0} = 1 \\ x_2(0) = 0 \end{cases}$$

R.: $\boldsymbol{x} = \begin{bmatrix} -2/15 \\ -6/15 \end{bmatrix} e^{-t} + \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -9/20 \\ 3/20 \end{bmatrix} e^{4t} + \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$

(9) $y_{n+1} + 3y_n = n, \quad y_0 = 1$

R.: $y_n = \dfrac{17}{16}(-3)^n - \dfrac{1}{16} + \dfrac{n}{4}$

(10) $y_{n+2} - 2y_{n+1} + 2y_n = 0, \quad y_0 = 1, \quad y_1 = 1$

R.: $y_n = 2^{n/2}\,\mathrm{sen}\left(\dfrac{\pi}{4}n\right)$

(11) $y_{n+2} - 3y_{n+1} + 2y_n = -1, \quad y_0 = 2, \quad y_1 = 4$

R.: $y_n = n + 1 + 2^n$

3 – Séries

Séries numéricas

$$\boxed{\sum_{r=1}^{\infty} w_r = w_1 + w_2 + w_3 + \ldots + w_n + \ldots}$$

$$\begin{tikzcd}
\sum_{r=1}^{\infty} w_r \arrow[r, "\text{convergente}"] \arrow[dr, "\text{divergente}"'] & S = \sum_{r=1}^{\infty} w_r = \underbrace{\sum_{r=1}^{n} w_r}_{\substack{S_n \\ \text{(soma)}}} + \underbrace{\sum_{r=n+1}^{\infty} w_r}_{\substack{R_n \\ \text{(erro de truncamento)}}} \\
& \text{O jeito é tocar um tango argentino!}
\end{tikzcd}$$

Teorema: Se a seqüência $\{w_r\}$ converge para α, então

$$\sum_{r=1}^{\infty} \left(w_r - w_{r+1} \right) = w_1 - \alpha$$

Teorema: Se a seqüência $\{w_r\}$ é divergente, então

(a) $\sum_{r=1}^{\infty} \left(w_{r+1} - w_r \right)$ é divergente;

(b) $\sum_{r=1}^{\infty} \left(\frac{1}{w_r} - \frac{1}{w_{r+1}} \right) = \frac{1}{w_1} \quad (w_r \neq 0)$.

Testes para verificação da convergência de séries

Teste da comparação: séries-padrão

Série geométrica: $\sum_{r=1}^{\infty} ak^{r-1}$ $\begin{cases} |k| < 1, & \text{convergente} \\ |k| \geq 1, & \text{divergente} \end{cases}$

Série "*p*": $\sum_{r=1}^{\infty} \frac{1}{r^{p}}$ $\begin{cases} p > 1, & \text{convergente} \\ p \leq 1, & \text{divergente} \end{cases}$

Convergência e erro de truncamento das séries positivas

Teste	*Erro de truncamento*
$\lim_{b\to\infty} \int_{1}^{b} w(x)\,dx$ existe (teste da integral)	$R_n < \lim_{b\to\infty} \int_{n}^{b} w(x)\,dx$
$\lim_{r\to\infty} \frac{w_{r+1}}{w_r} < 1$ (teste da razão)	$R_n < \frac{w_{n+1}}{1-L'}$ $L' = \frac{w_{n+1}}{w_n} < 1$
$\lim_{r\to\infty} w_r^{1/r} < 1$ (teste da raiz)	$R_n < \frac{w_{n+1}}{1-w_{n+1}^{1/(n+1)}}$

Erro de truncamento na série alternada convergente, $\sum_{r=1}^{\infty} (-1)^{r+1} a_r$

$$|R_n| < a_{n+1}$$

Série de Exercícios n^o 4 – Séries numéricas

Referência: *W. Kaplan, "Advanced Calculus", Addison-Wesley, Cap. 6 (1969).*

(1) Prove a convergência: ensaio da comparação.

$$\sum_{n=2}^{\infty}\frac{1}{n^3-1} \qquad \sum_{n=1}^{\infty}\frac{\text{sen}(n)}{n^2}$$

(2) Prove a divergência: ensaio da comparação.

$$\sum_{n=1}^{\infty}\frac{n+5}{n^2-3n-5} \qquad \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,\ln(n)}$$

(3) Prove a convergência: ensaio da integral.

$$\sum_{n=1}^{\infty}\frac{1}{n^2+1} \qquad \sum_{n=2}^{\infty}\frac{1}{n\ln^2(n)}$$

(4) Prove a convergência: ensaio da razão.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n!} \qquad \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(5) Prove a convergência: ensaio da raiz.

$$\sum_{n=1}^{\infty}\frac{1}{n^n} \qquad \sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$$

(6) Prove a convergência: ensaio das séries alternadas.

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)} \qquad \sum_{n=2}^{\infty}\frac{(-1)^n\ln(n)}{n}$$

(7) Calcule a soma com três algarismos significativos exatos.

7.1) $\sum_{n=1}^{\infty}\frac{1}{n!}$ **R.:** 1,71<u>8</u>

7.2) Função de Bessel de 1ª espécie e ordem zero no ponto $x=1$

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots$$ **R.:** 0.765<u>6</u>

7.3) $\int_0^1 \operatorname{sen}(x)dx=1-\frac{1}{3\cdot 3!}+\frac{1}{5\cdot 5!}-\frac{1}{7\cdot 7!}+\frac{1}{9\cdot 9!}-\ldots$ **R.:** 0.946<u>1</u>

7.4) Integral elíptica de 1ª espécie: $K=F(k,\pi/2)=\int_0^{\pi/2}\frac{d\theta}{\left(1-k^2\operatorname{sen}^2(\theta)\right)^{1/2}}$

$$K=F(0{,}5;\pi/2)=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 0{,}5^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 0{,}5^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 0{,}5^6+\ldots\right\}$$

Séries de Potências

Desenvolvimento em série de Taylor de $f(x)$ nas vizinhanças de $x = a$:

$$\boxed{f(x) = \sum_{k=1}^{\infty} c_k (x-a)^k}, \qquad c_k = \frac{f^{(k)}(a)}{k!}$$

Raio de convergência da série: $R = \lim\limits_{k\to\infty} \left| \dfrac{c_k}{c_{k+1}} \right|$

Erro de truncamento: $R_{n+1} = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \; a < \xi < x$

(Fórmula de Lagrange)

Série nº 5: Séries de Potências

(1) Desenvolver f em série de Taylor nas vizinhanças de a e determinar a região de convergência da série.

a) $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \quad a = 0$

R.: $\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \ldots \quad (-1 < x < 1)$

b) $f(x) = \dfrac{1}{x}, \quad a = 1$

R.: $\dfrac{1}{x} = \sum\limits_{n=0}^{\infty} (-1)^n (x-1)^n \quad (0 < x < 2)$

(2) Determinar a região de convergência das séries:

a) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots$ **R.:** $-1<x<1$

b) $\sec(x)=1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\ldots$ **R.:** $|x|<\frac{\pi}{2}$

c) $e^{\tan(x)}=1+x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{3x^4}{8}+\ldots$ **R.:** $|x|<\frac{\pi}{2}$

d) $(1-x)^{-1}=1-x+x^2-x^3+x^4-\ldots$ **R.:** $-1<x<1$

(3) Resolver $y''-2xy'+4xy=x^2+2x+2$ com o auxílio da série $y=\sum_{n=0}^{\infty}c_n x^n$.

R.:

$$y(x)=c_0\left(1-\frac{2}{3}x^3-\frac{2}{45}x^6-\frac{2}{405}x^9-\ldots\right)+c_1\left(x-\frac{1}{6}x^4-\frac{1}{63}x^7-\ldots\right)$$
$$+x^2+\frac{1}{3}x^3+\frac{1}{12}x^4+\frac{1}{45}x^6+\frac{1}{126}x^7+\frac{1}{405}x^9+\frac{1}{1134}x^{10}+\ldots$$

(Ayres, prob. 7, p. 204)

(4) Resolver em série de potências nas vizinhanças de $a=0$:

$$\int\frac{x}{(1+x)^{1/3}}dx\ .$$

R.:

$$\int\frac{x}{(1+x)^{1/3}}dx=\frac{1}{2}x^2-\frac{1}{3\cdot3}x^3+\frac{1\cdot4}{3\cdot6\cdot4}x^4-\frac{1\cdot4\cdot7}{3\cdot6\cdot9\cdot5}x^5+\frac{1\cdot4\cdot7\cdot10}{3\cdot6\cdot9\cdot12\cdot6}x^6-\ldots$$

(5) Calcular $\int_0^1 \frac{x}{(1+x)^{1/3}}\,dx$ com duas casas decimais exatas.

R.: $\int_0^1 \frac{x}{(1+x)^{1/3}}\,dx = 0{,}42\underline{3}6 \pm 0{,}0028$

(6) Calcular $\operatorname{erf}(1)$ com duas casas decimais exatas, sendo a <u>função erro</u> definida do modo

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du$$

Sugestão: $e^t = \sum_{r=0}^{\infty} \frac{t^r}{r!}$

R.: $\operatorname{erf}(1) = 0{,}8434 \pm 0{,}0008$

(7) Quantos termos são necessários para o cálculo de

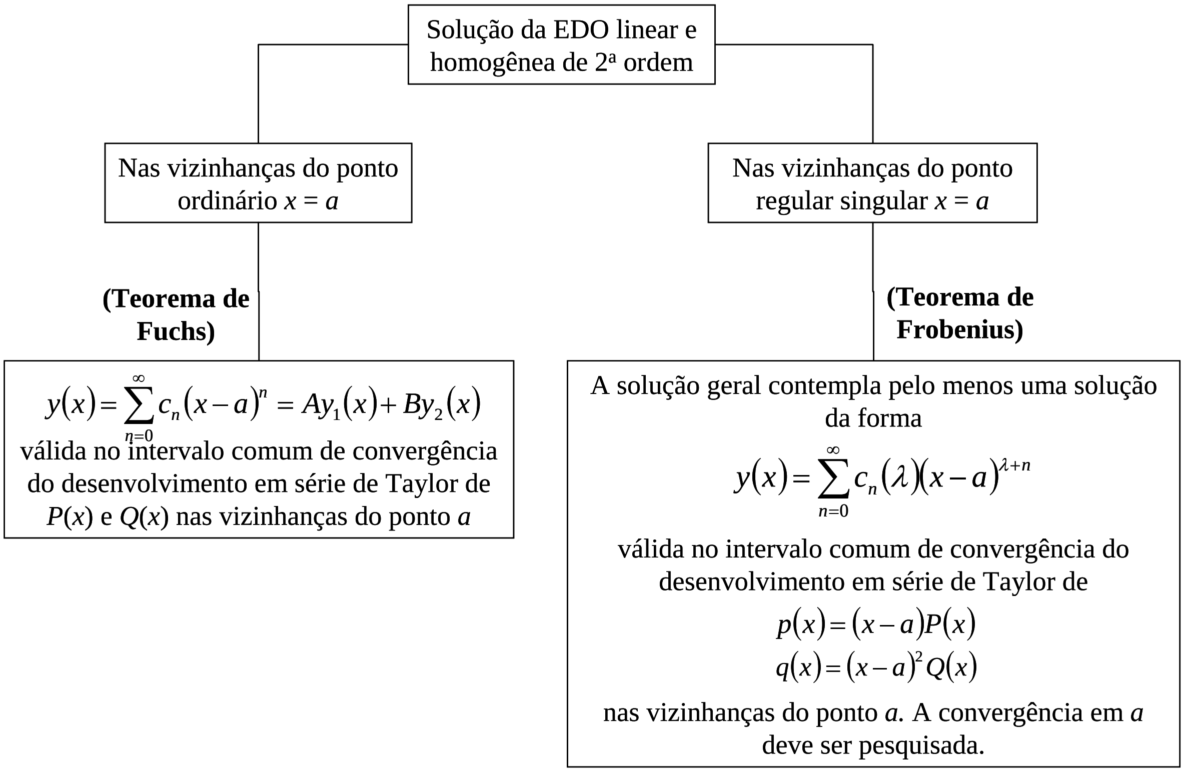
$$\sqrt{2\pi}\,\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left[11{,}9 + (n-1)\pi\right]^{1/2}}$$

com erro inferior, em módulo, a 10^{-2}?

R.: 2×10^4 termos.

4 – Solução em série de potências da EDO de 2^a ordem

$$y''+P(x)y'+Q(x)y=0$$



Solução da EDO linear e homogênea de 2^a ordem

Nas vizinhanças do ponto ordinário $x = a$

(Teorema de Fuchs)

$$y(x)=\sum_{n=0}^{\infty}c_n(x-a)^n = Ay_1(x)+By_2(x)$$

válida no intervalo comum de convergência do desenvolvimento em série de Taylor de $P(x)$ e $Q(x)$ nas vizinhanças do ponto a

Nas vizinhanças do ponto regular singular $x = a$

(Teorema de Frobenius)

A solução geral contempla pelo menos uma solução da forma

$$y(x)=\sum_{n=0}^{\infty}c_n(\lambda)(x-a)^{\lambda+n}$$

válida no intervalo comum de convergência do desenvolvimento em série de Taylor de

$$p(x)=(x-a)P(x)$$
$$q(x)=(x-a)^2Q(x)$$

nas vizinhanças do ponto a. A convergência em a deve ser pesquisada.

Equação	**Solução**	**Polinômios**
Euler $\boxed{x^2y''+axy'+by=0}$ Transformação: $x=e^t$ $\frac{d^2y}{dt^2}+(a-1)\frac{dy}{dt}+by=0$	$y(x)=Ax^{m_1}+Bx^{m_2}$ $m_1\neq m_2$ são raízes de $m^2+(a-1)m+b=0$	—
Bessel $\boxed{x^2y''+xy'+(\lambda^2x^2-n^2)y=0}$	$y(x)=AJ_n(\lambda x)+BY_n(\lambda x)$ $x\neq 0$	—
Legendre $\boxed{(1-x^2)y''-2xy'+n(n+1)y=0}$ $n=0,1,2,3,\ldots$	$y(x)=AU_n(x)+BV_n(x)$ $U_n(x)=1-\frac{n(n-1)}{2!}x^2+\frac{n(n-2)(n+1)(n+3)}{4!}x^4-\ldots$ $V_n(x)=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5-\ldots$ $-1<x<1$	$P_n(x)=\frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n$ $P_0(x)=1$, $P_1(x)=x$ $P_2(x)=\frac{1}{2}(3x^2-1)$
Chebyshev $\boxed{(1-x^2)y''-xy'+n^2y=0}$ $n=0,1,2,3,\ldots$	$y(x)=A\cdot F\left(\frac{n}{2},\frac{-n}{2},\frac{1}{2},x^2\right)+B\cdot F\left(\frac{1+n}{2},\frac{1-n}{2},\frac{3}{2},x^2\right)$ $-1<x<1$	$T_n(x)=\cos(n\cdot\cos^{-1}(x))$ $T_0(x)=1$, $T_1(x)=x$, $T_2(x)=2x^2-1$
Laguerre $\boxed{xy''+(1-x)y'+ny=0}$ $n=0,1,2,3,\ldots$	Solução em série de potências: $y_1(x)=F(-n,1,\infty)$	$L_n(x)=e^x\frac{d^n}{dx^n}(x^ne^{-x})$ $L_0(x)=1$, $L_1(x)=-x+1$, $L_2(x)=x^2-4x+2$
Hermite $\boxed{y''-2xy'+2ny=0}$ $n=0,1,2,3,\ldots$	$y(x)=A\cdot F\left(\frac{-n}{2},\frac{1}{2},x^2\right)+B\cdot F\left(\frac{1-n}{2},1,x^2\right)$	$H_n(x)=(-1)^ne^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$ $H_0(x)=1$, $H_1(x)=2x$, $H_2(x)=4x^2-2$

(ver definição das funções utilizadas nas próximas páginas)

Funções de Bessel de Primeira Classe de Ordem n:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\, \Gamma(n+k+1)} \qquad J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\, \Gamma(k+1-n)}$$

Funções de Bessel de Segunda Classe de Ordem n:

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\operatorname{sen}(n\pi)} & n \neq 0,1,2,\ldots \\ \dfrac{2}{\pi}\left[\ln(x/2) + \gamma\right] J_n(x) - \dfrac{1}{\pi}\displaystyle\sum_{k=0}^{n-1} (n-k-1)!\,(x/2)^{2k-n} - \dfrac{1}{\pi}\displaystyle\sum_{k=0}^{\infty} (-1)^k \left[\Phi(k) + \Phi(n+k)\right] \dfrac{(x/2)^{2k+n}}{k!\,(n+k)!} & n = 0,1,2,\ldots \end{cases}$$

em que $\gamma = 0{,}5775156\ldots$ é a constante de Euler e $\Phi(k) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k}$, $\Phi(0) = 0$.

Funções Modificadas de Bessel de Primeira Classe de Ordem n:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!\, \Gamma(n+k+1)} \qquad I_{-n}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k-n}}{k!\, \Gamma(k+1-n)}$$

Funções Modificadas de Bessel de Segunda Classe de Ordem n:

$$K_n(x)=\begin{cases}\dfrac{\pi\left[I_{-n}(x)-I_n(x)\right]}{2\,\mathrm{sen}(n\pi)} & n\neq 0,1,2,\ldots\\ (-1)^{n+1}\left[\ln(x/2)+\gamma\right]I_n(x)+\dfrac{1}{2}\displaystyle\sum_{k=0}^{n-1}(-1)^k(n-k-1)!\,(x/2)^{2k-n}+ & \\ \qquad +\dfrac{(-1)^n}{2}\displaystyle\sum_{k=0}^{\infty}\left[\Phi(k)+\Phi(n+k)\right]\dfrac{(x/2)^{2k+n}}{k!(n+k)!} & n=0,1,2,\ldots\end{cases}$$

Função Hipergeométrica: $F(\alpha,\beta,\gamma,x)=1+\sum_{n=1}^{\infty}\dfrac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}x^n$

Função Hipergeométrica Confluente: $F(\alpha,\gamma,x)=1+\sum_{n=1}^{\infty}\dfrac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}x^n$

Soluções generalizadas de famílias de Equações Diferenciais Ordinárias de Segunda Ordem

$f(\nabla)y + x^k g(\nabla)y = 0$, em que o operador ∇ é definido por $\nabla y = xy'$	
Família	**Solução**
Bessel $\boxed{\begin{array}{l}(\nabla - a)(\nabla - b)y + p^2 x^{2m} y = 0 \\ (\nabla - a)(\nabla - b)y - p^2 x^{2m} y = 0\end{array}}$	$y(x) = x^{\left(\frac{a+b}{2}\right)}\left[AJ_\nu\left(\frac{px^m}{m}\right) + BY_\nu\left(\frac{px^m}{m}\right)\right],\ \nu = \frac{b-a}{2m}\ (m \neq 0)$ $y(x) = x^{\left(\frac{a+b}{2}\right)}\left[AI_\nu\left(\frac{px^m}{m}\right) + BK_\nu\left(\frac{px^m}{m}\right)\right],\ \nu = \frac{b-a}{2m}\ (m \neq 0)$ $x \neq 0$
Hipergeométrica Confluente $\boxed{(\nabla + a)(\nabla + b)y - qx^p(\nabla + c)y = 0}$ $p \neq 0$	$y(x) = Ax^{-a}F\left(\alpha, \gamma, \frac{qx^p}{p}\right) + Bx^{-a}\left(\frac{qx^p}{p}\right)^{1-\gamma} F\left(\alpha - \gamma + 1, 2 - \gamma, \frac{qx^p}{p}\right)$ $\alpha = \frac{c-a}{p},\ \gamma = \frac{b-a}{q} + 1$
Hipergeométrica $\boxed{(\nabla + a)(\nabla + b)y - qx^p(\nabla + c)(\nabla + d)y = 0}$ $p \neq 0$	Nas vizinhanças de $x = 0$: $y(x) = Ax^{-a}F\left(\frac{c-a}{p}, \frac{d-a}{p}, \frac{b-a}{p} + 1, qx^p\right)$ $+ Bx^{-a}\left(qx^p\right)^{-\frac{(b-a)}{p}} F\left(\frac{c-b}{p}, \frac{d-b}{p}, 1 - \frac{b-a}{p}, qx^p\right)$ Para $\lvert x \rvert > 1$ $y(x) = Ax^{-a}\left(qx^p\right)^{\frac{(a-c)}{p}} F\left(\frac{c-a}{p}, \frac{c-b}{p}, \frac{c-d}{p} + 1, \left(qx^p\right)^{-1}\right)$ $+ Bx^{-a}\left(qx^p\right)^{-\frac{(a-d)}{p}} F\left(\frac{d-a}{p}, \frac{d-b}{p}, \frac{d-c}{p} + 1, \left(qx^p\right)^{-1}\right)$

Série de Exercícios n^o 6 – Solução de EDO linear de 2^a ordem em série de potências.

(1) Resolver nas vizinhanças de $x = 0$

$$\begin{cases} y''+xy'+\ln(x+1)y = 0 \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

R.: $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad -1 < x \le 1$

Solução geral: $y(x) = c_0\left(1 - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\right) + c_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \ldots\right) \quad -1 < x \le 1$

Solução particular: $y(x) = x - \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{20} + \frac{x^6}{180} - \ldots \quad -1 < x \le 1$

(2) Resolver nas vizinhanças de $x = 1$

$$\begin{cases} x^2y''-2xy'+\ln(x)y = 0 \\ y(1) = y'(1) = 1/2 \end{cases}$$

Sugestão: faça $t = x - 1$

R.: $\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots \quad 0 < x \le 1$

$y(x) = \frac{1}{2} + \frac{1}{2}(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{48}(x-1)^4 + \ldots \quad 0 < x \le 1$

(3) Resolver nas vizinhanças de $x = 0$ e fornecer o intervalo de convergência da resposta

$$\begin{cases} y''+3xy'+e^x y = 2x \\ y(0) = 1 \quad y'(0) = -1 \end{cases}$$

R.: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $-\infty < x < \infty$

$y(x) = 1 - x - \frac{x^2}{2} + \frac{5x^3}{16} + \frac{x^4}{3} - \ldots \quad -\infty < x < \infty$

(4) Resolver a equação de Euler: $x^2 y'' + xy' - y = 1$

R.: $x = e^t$, $y(x) = Ax + Bx^{-1} - 1$

(5) Resolver nas vizinhanças de *x* = 0:

a) $\begin{cases} x^{1/2} y'' + \lambda y = 0 \\ y(0) = 0 \end{cases}$ R.: $\nabla(\nabla - 1)y + \lambda x^{3/2} y = 0$

b) $\begin{cases} x^2 y'' + 2xy' - (9x^3 + 2)y = 0 \\ \lim\limits_{x\to\infty} y(x) = 0 \end{cases}$ R.: $(\nabla - 1)(\nabla + 2)y - 9x^3 y = 0$

c) $\begin{cases} x^4 y'' + x^3 y' - (x^2 + 1)y = 0 \\ y(0) = 0 \end{cases}$ R.: $(\nabla^2 - 1)y - \frac{1}{x^2} y = 0$

d) $\begin{cases} y'' + \frac{1}{x} y' - \frac{1-x}{x^2} y = 0 \\ y(0) = 0 \end{cases}$ R.: $(\nabla + 1)(\nabla - 1)y + xy = 0$

e) $2xy'' + (x+1)y' + 3y = 0$ R.: $\nabla\left(\nabla - \frac{1}{2}\right)y + \frac{1}{2}x(\nabla + 3)y = 0$

f) $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' - \frac{1}{4} y = 0$

R.: $\nabla\left(\nabla + \frac{1}{2}\right)y - x\left(\nabla + \frac{1}{2}\right)\left(\nabla + \frac{1}{2}\right)y = 0$

g) $\begin{cases} xy''-y'+4x^3y=0 \\ y(0)=0,\ y'\left(\sqrt{\dfrac{\pi}{2}}\right)=1 \end{cases}$ R.: $\nabla(\nabla-2)y+4x^4y=0$

h) $x^2y''+xy'+\left(x^2-\dfrac{1}{16}\right)y=0$ R.: $\left(\nabla^2-\dfrac{1}{16}\right)y+x^2y=0$

i) $y''-y'-\dfrac{y}{x}=0$ R.: $\nabla(\nabla-1)y-x(\nabla+1)y=0$

(6) Resolva o problema de condução de calor na aleta triangular:

$$\begin{cases} x^2y''+xy'-\Omega^2\ xy=0 \\ y(1)=T_p-T_a\ \text{, constante} \\ y(0)\ \text{é finito} \end{cases}$$

R.: $\nabla^2y-\Omega^2\ xy=0$

$y(x)=\left(T_p-T_a\right)\dfrac{I_0\left(2\Omega\, x^{1/2}\right)}{I_0\left(2\Omega\, L^{1/2}\right)}$

5 – Problema de Sturm-Liouville Homogêneo de 2^a ordem

Forma da EDO linear e homogênea de 2^a ordem:

$$\boxed{X''+g_1(x)X'+\left[g_2(x)+\mu g_3(x)\right]X=0}$$

$$\boxed{\left[s(x)X'(x)\right]'+\left[q(x)+\mu p(x)\right]X(x)=0}$$

onde

$$s(x)=\exp\int g_1(x)dx\,,\qquad q(x)=g_2(x)s(x),\qquad p(x)=g_3(x)s(x)$$

O Problema de Sturm-Liouville

$$\boxed{\begin{cases}\left[s(x)X'(x)\right]'+\left[q(x)+\mu p(x)\right]X(x)=0\\ k_1X(a)+k_2X'(a)=0\\ l_1X(b)+l_2X'(b)=0\end{cases}}$$

Teorema 1: os valores característicos são reais e formam uma seqüência infinita

$$\mu_1<\mu_2<\mu_3<\ldots<\mu_n<\ldots$$

sendo $\lim_{n\to\infty}\mu_n$ infinito.

Teorema 2: as funções características $X_m(x)$ e $X_n(x)$ correspondentes a μ_m e μ_n são ortogonais em relação à função peso $p(x)$ em $[a,b]$, isto é

$$\int_a^b p(x)X_m(x)X_n(x)dx=0\,,\quad m\neq n$$

Desenvolvimento de $f(x)$ em série de funções ortogonais $\{\phi(x)\}$ em relação à função peso $p(x)$ em (a,b)

Teorema 3: sejam $\phi_1, \phi_2, \phi_3, ...\phi_n, ...$ as funções características do problema de Sturm-Liouville. Sejam f e f' contínuas por partes no intervalo $a \le x \le b$. Então a série

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x), \qquad A_n = \frac{\int_a^b p(x) f(x) \phi_n(x) dx}{\int_a^b p(x) \phi_n{}^2(x) dx}$$

converge para $\dfrac{f(x+) + f(x-)}{2}$ em cada ponto do intervalo aberto $a < x < b$.

OBS.: $f(x+) = \lim\limits_{x \to x+} f(x)$ (limite de f à direita no ponto x)

$f(x-) = \lim\limits_{x \to x-} f(x)$ (limite de f à esquerda no ponto x)

Para funções contínuas, $f(x+) = f(x-)$.

Séries de Fourier: muitas funções podem também ser representadas como séries de Fourier, as quais são formadas por combinações lineares de senos e cossenos.

A seqüência

$$\boxed{\left\{ \text{sen}\,\frac{n\pi\, x}{c}, \cos\frac{n\pi\, x}{c} \right\},\ n = 0,1,2,...}$$

é um conjunto de funções ortogonais em $(d, d+2c)$ em relação à função peso $p(x) = 1$.

A série de Fourier de $f(x)$, definida em $(d, d+2c)$, é dada por

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \operatorname{sen}\frac{n\pi x}{c}\right)}$$

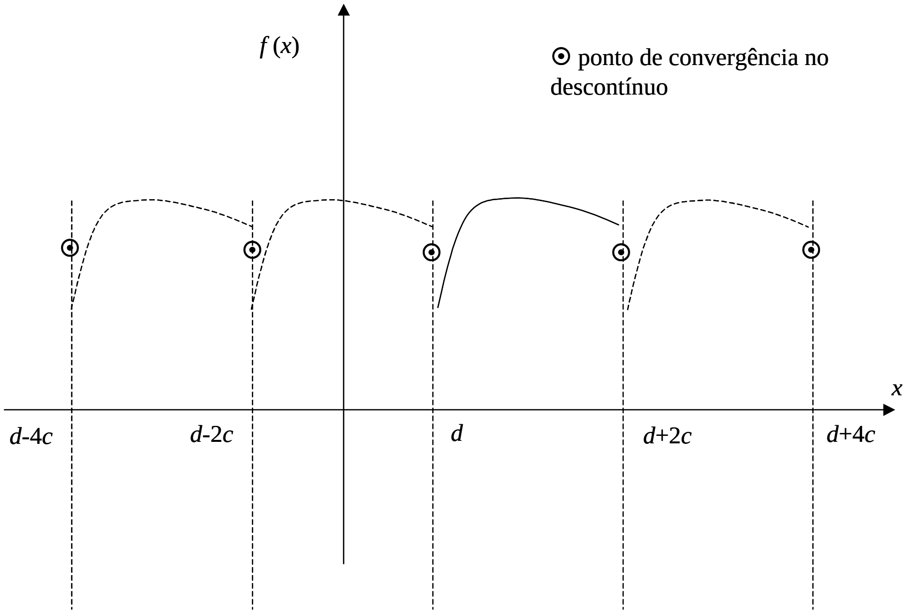
Em que

$$a_0 = \frac{1}{2c}\int_d^{d+2c} f(x)\,dx$$

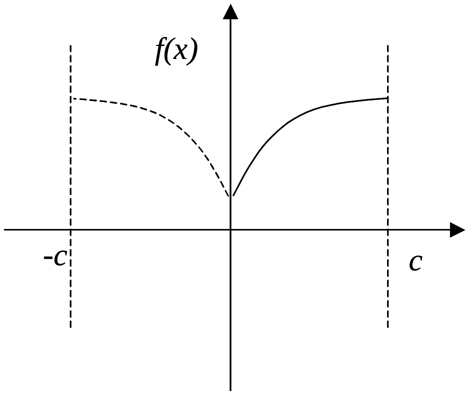
$$a_n = \frac{1}{c}\int_d^{d+2c} f(x)\cos\frac{n\pi x}{c}\,dx\,, \quad n = 1,2,3,\ldots$$

$$b_n = \frac{1}{c}\int_d^{d+2c} f(x)\operatorname{sen}\frac{n\pi x}{c}\,dx\,, \quad n = 1,2,3,\ldots$$

A série de Fourier de uma função contínua por partes (como a função periódica de período 2c abaixo) também converge para o valor $\frac{f(x+)+f(x-)}{2}$ nos pontos de descontinuidade.



Série de Fourier da expansão <u>par</u> de $f(x)$*, definida em* $(-c,c)$*:*



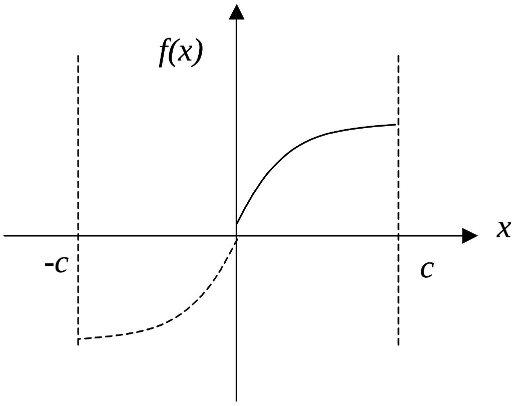
$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c}\right)}$$

Em que

$$a_0 = \frac{1}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx\ , \quad n = 1,2,3,...$$

Série de Fourier da expansão <u>ímpar</u> de $f(x)$*, definida em* $(-c,c)$*:*



$$\boxed{f(x) = \sum_{n=1}^{\infty}\left(b_n \operatorname{sen}\frac{n\pi x}{c}\right)}$$

Em que

$$b_n = \frac{2}{c}\int_0^c f(x)\operatorname{sen}\frac{n\pi x}{c}\,dx\ , \quad n = 1,2,3,...$$

Série nº 7 – Problema de Sturm-Liouville

(1) Séries de Fourier

a) Desenvolver $f(t)=\begin{cases} t, & 0\le t\le 1 \\ 2-t, & 1\le t\le 2 \end{cases}$ em série só de senos e em série só de cossenos.

R.: $f(t)=\dfrac{8}{\pi^2}\displaystyle\sum_{n=0}^{\infty}\left(\dfrac{(-1)^n}{(2n+1)^2}\operatorname{sen}\dfrac{(2n+1)\pi t}{2}\right)$

$$f(t)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}\cos(2n+1)\pi t\right)$$

a) Desenvolver $f(t)=t^2\,,\quad -\pi\le t\le \pi$

R.: $f(t)=\dfrac{\pi^2}{3}+4\displaystyle\sum_{n=1}^{\infty}\left(\dfrac{(-1)^n\cos(nt)}{n^2}\right)$

(2) Problema de Sturm-Liouville:

$$\begin{cases} xy''-3y'+\lambda^2 xy=0 \\ y(0)=0\,,\ y(1)=0 \end{cases}$$

a) Mostrar que os valores característicos são as raízes de $J_2(\lambda_n)=0$, isto é

n	1	2	3	4	...
λ_n	5,1356	8,4172	11,6198	14,7960	...

b) Mostrar que as funções características são $y_n(x)=x^2 J_2(\lambda_n x)$.

c) Mostrar que a função peso é $p(x)=\dfrac{1}{x^3}$.

(3) Determinar os valores característicos λ_n do problema de Sturm-Liouville

$$\begin{cases} y''+a^2(\lambda - x)^2\, y = 0 \\ y(0) = 0,\ y'(\lambda) = 0 \end{cases}$$

Sugestão: faça $t = \lambda - x$.

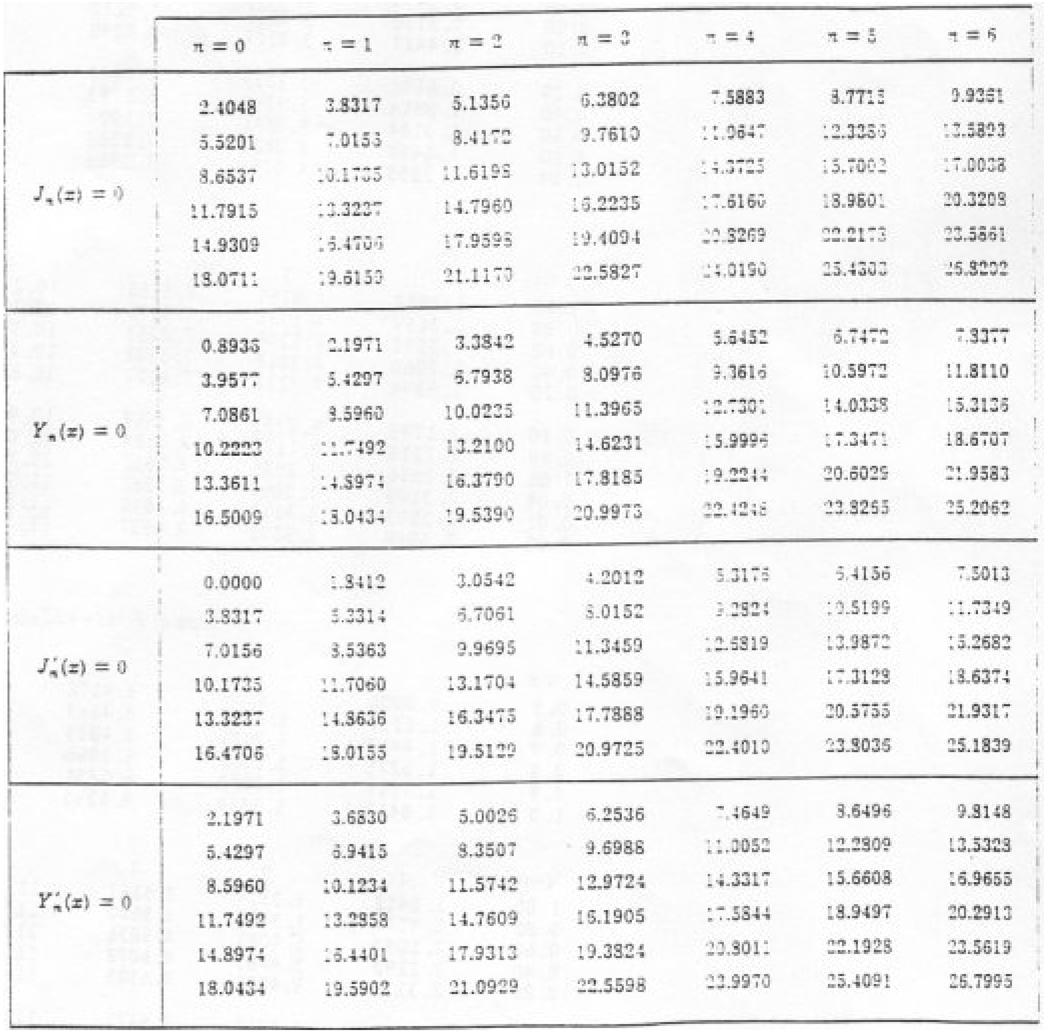
R.: raízes de $J_{1/4}\left(\frac{a\lambda_n{}^2}{2}\right) = 0$.

(4) Determinar os valores e funções característicos e a função peso do seguinte problema de Sturm-Liouville:

$$y''-2y'+\left(1+\lambda^2\right)y = 0$$
$$y(0) = 0,\ y(1) = 0$$

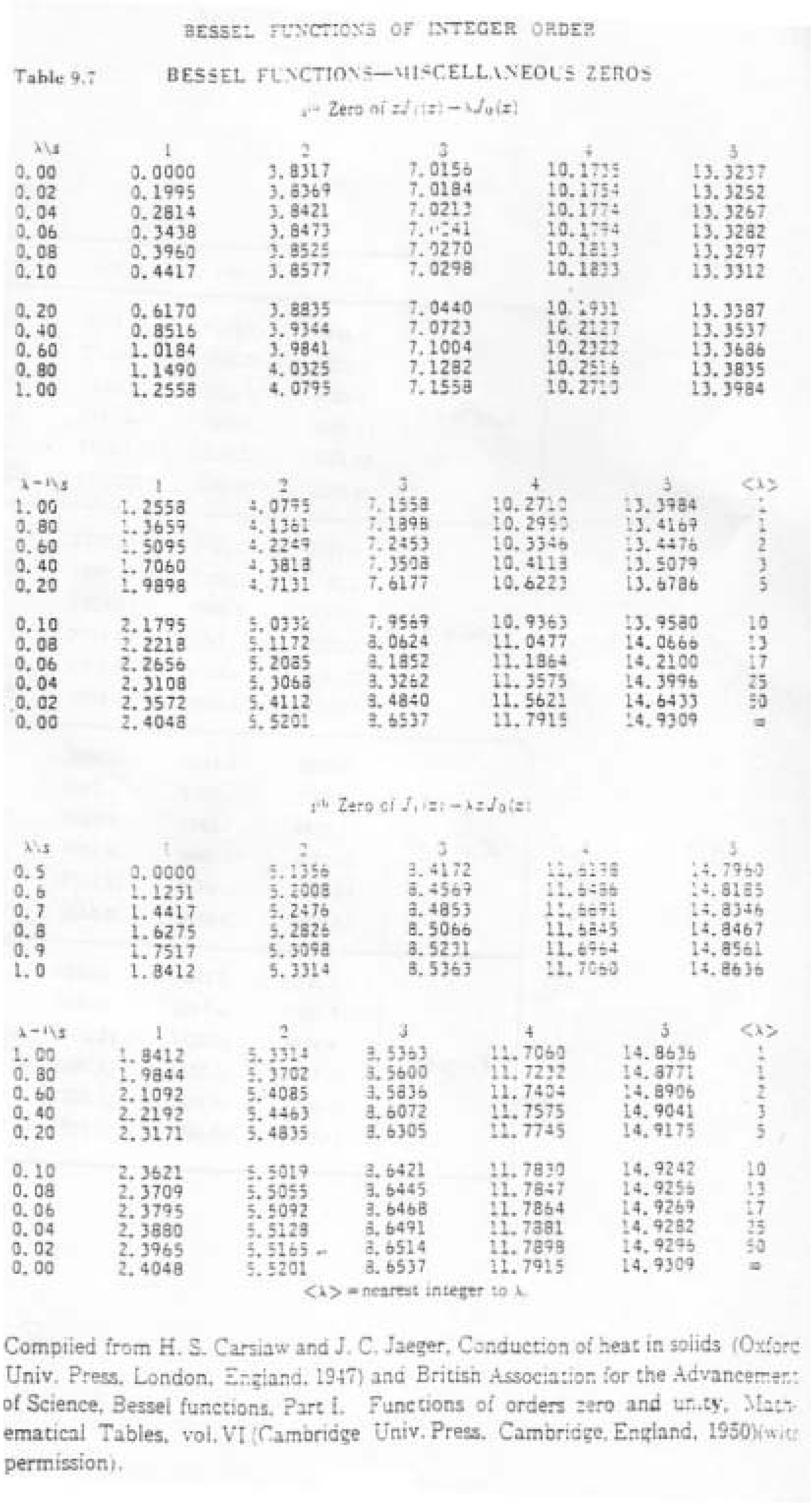
R.: $\lambda_n = n\pi \quad (n = 0,1,2,...), \quad y_n(x) = A_n e^x \operatorname{sen}(n\pi x), \quad p(x) = e^{-2x}$

Zeros das Funções de Bessel (Spiegel e Liu, p. 311)



	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$J_n(x) = 0$	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715	9.9361
	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386	13.5893
	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002	17.0038
	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801	20.3208
	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178	23.5861
	18.0711	19.6159	21.1170	22.5827	24.0190	25.4303	26.8202
$Y_n(x) = 0$	0.8936	2.1971	3.3842	4.5270	5.6452	6.7472	7.8377
	3.9577	5.4297	6.7938	8.0976	9.3616	10.5972	11.8110
	7.0861	8.5960	10.0235	11.3965	12.7301	14.0338	15.3136
	10.2223	11.7492	13.2100	14.6231	15.9996	17.3471	18.6707
	13.3611	14.8974	16.3790	17.8185	19.2244	20.6029	21.9583
	16.5009	18.0434	19.5390	20.9973	22.4248	23.8265	25.2062
$J'_n(x) = 0$	0.0000	1.8412	3.0542	4.2012	5.3176	6.4156	7.5013
	3.8317	5.3314	6.7061	8.0152	9.2824	10.5199	11.7349
	7.0156	8.5363	9.9695	11.3459	12.6819	13.9872	15.2682
	10.1735	11.7060	13.1704	14.5859	15.9641	17.3128	18.6374
	13.3237	14.8636	16.3475	17.7888	19.1960	20.5755	21.9317
	16.4706	18.0155	19.5129	20.9725	22.4010	23.8036	25.1839
$Y'_n(x) = 0$	2.1971	3.6830	5.0026	6.2536	7.4649	8.6496	9.8148
	5.4297	6.9415	8.3507	9.6988	11.0052	12.2809	13.5328
	8.5960	10.1234	11.5742	12.9724	14.3317	15.6608	16.9655
	11.7492	13.2858	14.7609	16.1905	17.5844	18.9497	20.2913
	14.8974	16.4401	17.9313	19.3824	20.8011	22.1928	23.5619
	18.0434	19.5902	21.0929	22.5598	23.9970	25.4091	26.7995

Abramovitz & Stegun: “Handbook of Mathematical Functions”, Dover, New York, p. 414



BESSEL FUNCTIONS OF INTEGER ORDER

Table 9.7 BESSEL FUNCTIONS—MISCELLANEOUS ZEROS

s^{th} Zero of $zJ_1(z) - \lambda J_0(z)$

$\lambda \backslash s$	1	2	3	4	5
0.00	0.0000	3.8317	7.0156	10.1735	13.3237
0.02	0.1995	3.8369	7.0184	10.1754	13.3252
0.04	0.2814	3.8421	7.0213	10.1774	13.3267
0.06	0.3438	3.8473	7.0241	10.1794	13.3282
0.08	0.3960	3.8525	7.0270	10.1813	13.3297
0.10	0.4417	3.8577	7.0298	10.1833	13.3312
0.20	0.6170	3.8835	7.0440	10.1931	13.3387
0.40	0.8516	3.9344	7.0723	10.2127	13.3537
0.60	1.0184	3.9841	7.1004	10.2322	13.3686
0.80	1.1490	4.0325	7.1282	10.2516	13.3835
1.00	1.2558	4.0795	7.1558	10.2710	13.3984

$\lambda^{-1} \backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.2558	4.0795	7.1558	10.2710	13.3984	1
0.80	1.3659	4.1361	7.1898	10.2950	13.4169	1
0.60	1.5095	4.2249	7.2453	10.3346	13.4476	2
0.40	1.7060	4.3818	7.3508	10.4113	13.5079	3
0.20	1.9898	4.7131	7.6177	10.6223	13.6786	5
0.10	2.1795	5.0332	7.9569	10.9363	13.9580	10
0.08	2.2218	5.1172	8.0624	11.0477	14.0666	13
0.06	2.2656	5.2085	8.1852	11.1864	14.2100	17
0.04	2.3108	5.3068	8.3262	11.3575	14.3996	25
0.02	2.3572	5.4112	8.4840	11.5621	14.6433	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

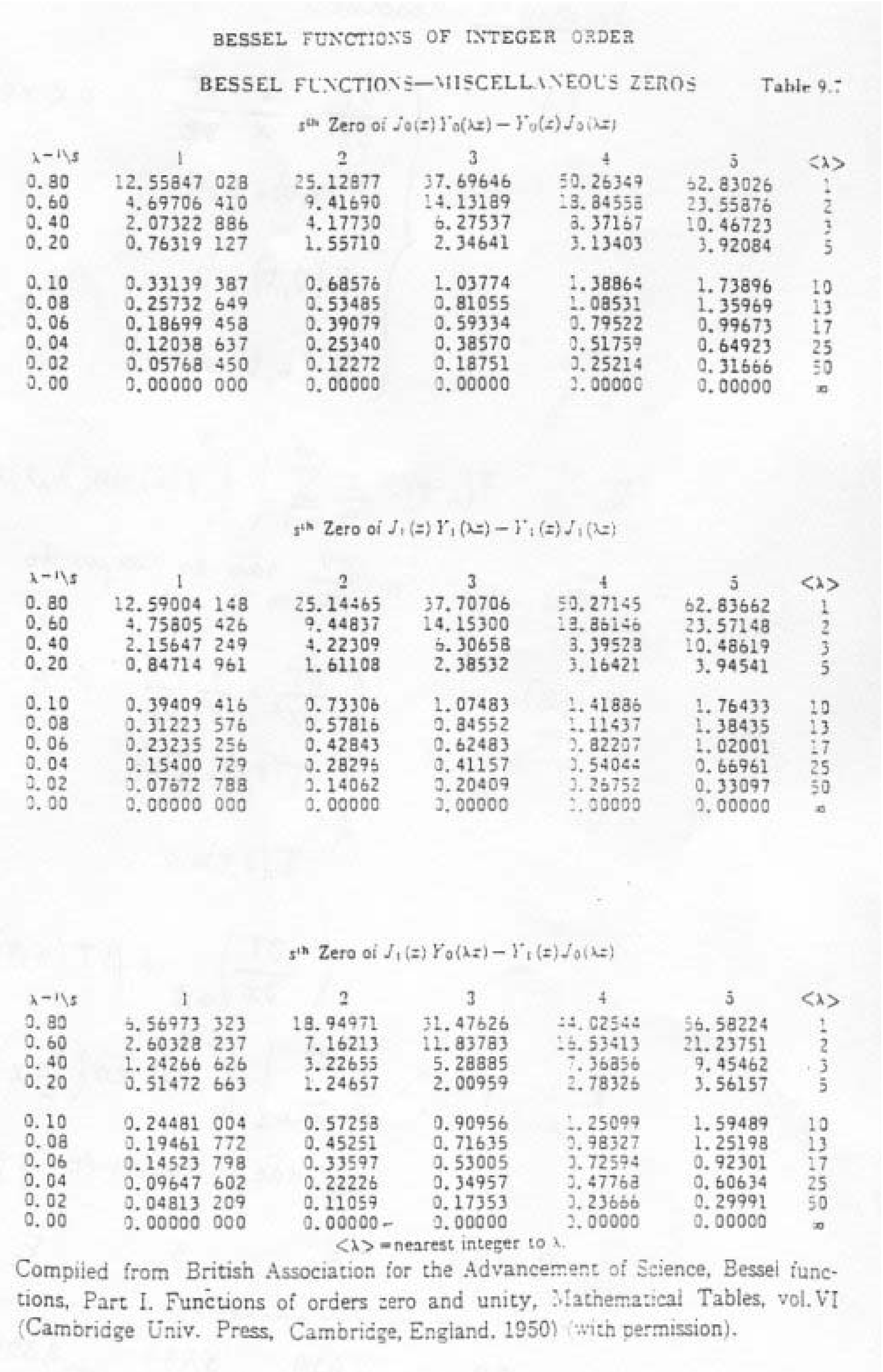
s^{th} Zero of $J_1(z) - \lambda z J_0(z)$

$\lambda \backslash s$	1	2	3	4	5
0.5	0.0000	5.1356	8.4172	11.6198	14.7960
0.6	1.1231	5.2008	8.4569	11.6436	14.8185
0.7	1.4417	5.2476	8.4853	11.6691	14.8346
0.8	1.6275	5.2826	8.5066	11.6845	14.8467
0.9	1.7517	5.3098	8.5231	11.6964	14.8561
1.0	1.8412	5.3314	8.5363	11.7060	14.8636

$\lambda^{-1} \backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.8412	5.3314	8.5363	11.7060	14.8636	1
0.80	1.9844	5.3702	8.5600	11.7232	14.8771	1
0.60	2.1092	5.4085	8.5836	11.7404	14.8906	2
0.40	2.2192	5.4463	8.6072	11.7575	14.9041	3
0.20	2.3171	5.4835	8.6305	11.7745	14.9175	5
0.10	2.3621	5.5019	8.6421	11.7830	14.9242	10
0.08	2.3709	5.5055	8.6445	11.7847	14.9256	13
0.06	2.3795	5.5092	8.6468	11.7864	14.9269	17
0.04	2.3880	5.5128	8.6491	11.7881	14.9282	25
0.02	2.3965	5.5165	8.6514	11.7898	14.9296	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

$\langle\lambda\rangle$ = nearest integer to λ.

Compiled from H. S. Carslaw and J. C. Jaeger, Conduction of heat in solids (Oxford Univ. Press, London, England, 1947) and British Association for the Advancement of Science, Bessel functions, Part I. Functions of orders zero and unity, Mathematical Tables, vol. VI (Cambridge Univ. Press, Cambridge, England, 1950) (with permission).



BESSEL FUNCTIONS OF INTEGER ORDER

BESSEL FUNCTIONS—MISCELLANEOUS ZEROS

Table 9.7

s^{th} Zero of $J_0(z)Y_0(\lambda z)-Y_0(z)J_0(\lambda z)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
0.80	12.55847 028	25.12877	37.69646	50.26349	62.83026	1
0.60	4.69706 410	9.41690	14.13189	18.84558	23.55876	2
0.40	2.07322 886	4.17730	6.27537	8.37167	10.46723	3
0.20	0.76319 127	1.55710	2.34641	3.13403	3.92084	5
0.10	0.33139 387	0.68576	1.03774	1.38864	1.73896	10
0.08	0.25732 649	0.53485	0.81055	1.08531	1.35969	13
0.06	0.18699 458	0.39079	0.59334	0.79522	0.99673	17
0.04	0.12038 637	0.25340	0.38570	0.51759	0.64923	25
0.02	0.05768 450	0.12272	0.18751	0.25214	0.31666	50
0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞

s^{th} Zero of $J_1(z)Y_1(\lambda z)-Y_1(z)J_1(\lambda z)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
0.80	12.59004 148	25.14465	37.70706	50.27145	62.83662	1
0.60	4.75805 426	9.44837	14.15300	18.86146	23.57148	2
0.40	2.15647 249	4.22309	6.30658	8.39528	10.48619	3
0.20	0.84714 961	1.61108	2.38532	3.16421	3.94541	5
0.10	0.39409 416	0.73306	1.07483	1.41886	1.76433	10
0.08	0.31223 576	0.57816	0.84552	1.11437	1.38435	13
0.06	0.23235 256	0.42843	0.62483	0.82207	1.02001	17
0.04	0.15400 729	0.28296	0.41157	0.54044	0.66961	25
0.02	0.07672 788	0.14062	0.20409	0.26752	0.33097	50
0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞

s^{th} Zero of $J_1(z)Y_0(\lambda z)-Y_1(z)J_0(\lambda z)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
0.80	6.56973 323	18.94971	31.47626	44.02544	56.58224	1
0.60	2.60328 237	7.16213	11.83783	16.53413	21.23751	2
0.40	1.24266 626	3.22655	5.28885	7.36856	9.45462	3
0.20	0.51472 663	1.24657	2.00959	2.78326	3.56157	5
0.10	0.24481 004	0.57258	0.90956	1.25099	1.59489	10
0.08	0.19461 772	0.45251	0.71635	0.98327	1.25198	13
0.06	0.14523 798	0.33597	0.53005	0.72594	0.92301	17
0.04	0.09647 602	0.22226	0.34957	0.47768	0.60634	25
0.02	0.04813 209	0.11059	0.17353	0.23666	0.29991	50
0.00	0.00000 000	0.00000 –	0.00000	0.00000	0.00000	∞

$\langle\lambda\rangle$ = nearest integer to λ.

Compiled from British Association for the Advancement of Science, Bessel functions, Part I. Functions of orders zero and unity, Mathematical Tables, vol. VI (Cambridge Univ. Press, Cambridge, England. 1950) (with permission).

6 – Solução da EDP por separação de variáveis

Série nº 8: resolver

(1)
$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}, \quad 0 < x < a,\ t > 0 \\ T(x,0) = f(x) \\ T(0,t) = 0 \\ T(a,t) = 0 \end{cases}$$

R.: $T(x,t) = \dfrac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a f(x)\operatorname{sen}(\lambda_n x)\,dx\right]\operatorname{sen}(\lambda_n x)\exp\left[-\alpha\lambda_n{}^2 t\right]$

- $\lambda_n = \dfrac{n\pi}{a}$ são as raízes de $\operatorname{sen}(\lambda_n a) = 0$.

(2)
$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}, \quad 0 < x < a,\ t > 0 \\ T(x,0) = f(x) \\ T(0,t) = 0 \\ \left.\dfrac{\partial T}{\partial x}\right|_{a,t} + \beta T(a,t) = 0 \end{cases}$$

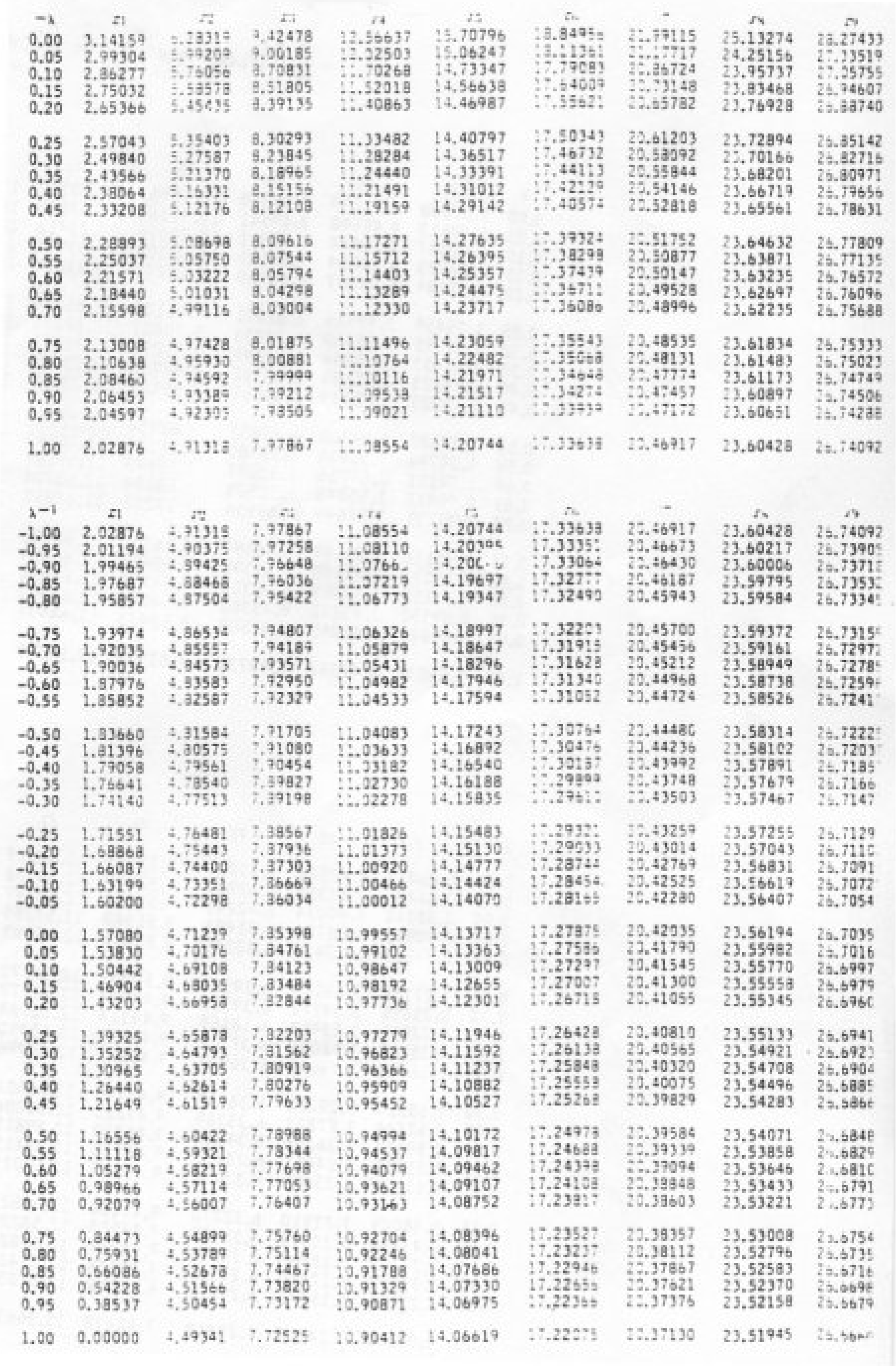
R.: $T(x,t) = \dfrac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a f(x)\operatorname{sen}\left(\dfrac{z_n}{a}x\right)dx\right]\operatorname{sen}\left(\dfrac{z_n}{a}x\right)\exp\left[-\alpha\left(\dfrac{z_n}{a}\right)^2 t\right]$

- z_n são as raízes de $\tan(z_n) = -\dfrac{z_n}{a\beta}$ (veja tabela na próxima página – Abramovitz e Stegun, 1965)

$\Omega = 1/a\beta$	z_1	z_2	z_3	z_4	z_5	z_6
0,2	2,654	5,454	8,391	11,41	14,47	17,56

$$\underbrace{\quad}_{2,800} \quad \underbrace{\quad}_{2,937} \quad \underbrace{\quad}_{3,019} \quad \underbrace{\quad}_{3,06} \quad \underbrace{\quad}_{3,09} \quad \to \pi$$

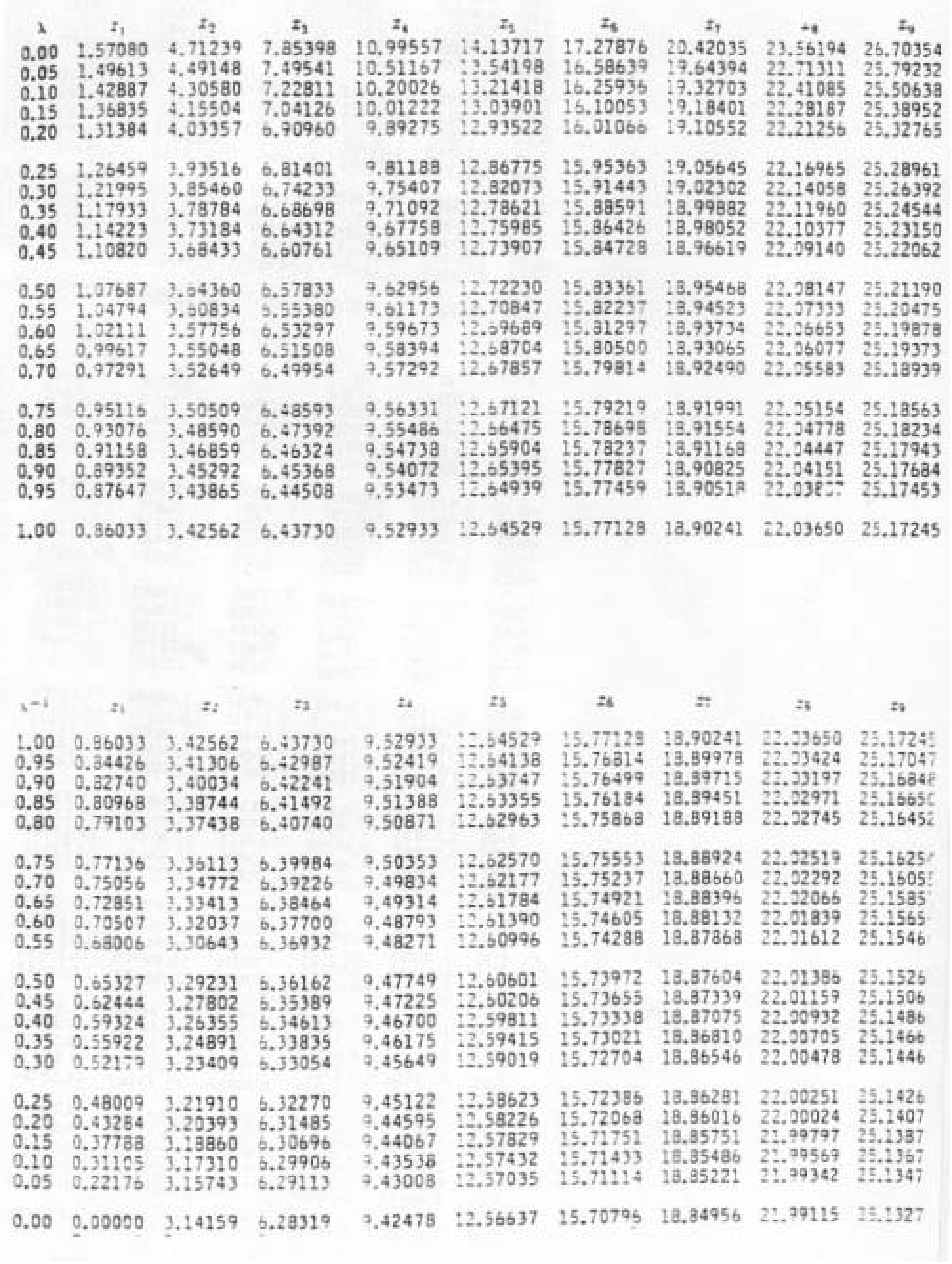
Raízes da Equação tg(x_n) = λx_n
(Abramovitz & Stegun p. 224)



$-\lambda$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0.00	3.14159	6.28319	9.42478	12.56637	15.70796	18.84956	21.99115	25.13274	28.27433
0.05	2.99304	5.99209	9.00185	12.02503	15.06247	18.11361	21.17717	24.25156	27.33519
0.10	2.86277	5.76056	8.70831	11.70268	14.73347	17.79083	20.86724	23.95737	27.05755
0.15	2.75032	5.58578	8.51805	11.52018	14.56638	17.64009	20.73148	23.83468	26.94607
0.20	2.65366	5.45435	8.39135	11.40863	14.46987	17.55621	20.65782	23.76928	26.88740
0.25	2.57043	5.35403	8.30293	11.33482	14.40797	17.50343	20.61203	23.72894	26.85142
0.30	2.49840	5.27587	8.23845	11.28284	14.36517	17.46732	20.58092	23.70166	26.82716
0.35	2.43566	5.21370	8.18965	11.24440	14.33391	17.44113	20.55844	23.68201	26.80971
0.40	2.38064	5.16331	8.15156	11.21491	14.31012	17.42129	20.54146	23.66719	26.79656
0.45	2.33208	5.12176	8.12108	11.19159	14.29142	17.40574	20.52818	23.65561	26.78631
0.50	2.28893	5.08698	8.09616	11.17271	14.27635	17.39324	20.51752	23.64632	26.77809
0.55	2.25037	5.05750	8.07544	11.15712	14.26395	17.38299	20.50877	23.63871	26.77135
0.60	2.21571	5.03222	8.05794	11.14403	14.25357	17.37439	20.50147	23.63235	26.76572
0.65	2.18440	5.01031	8.04298	11.13289	14.24475	17.36711	20.49528	23.62697	26.76096
0.70	2.15598	4.99116	8.03004	11.12330	14.23717	17.36086	20.48996	23.62235	26.75688
0.75	2.13008	4.97428	8.01875	11.11496	14.23059	17.35543	20.48535	23.61834	26.75333
0.80	2.10638	4.95930	8.00881	11.10764	14.22482	17.35068	20.48131	23.61483	26.75023
0.85	2.08460	4.94592	7.99999	11.10116	14.21971	17.34648	20.47774	23.61173	26.74749
0.90	2.06453	4.93389	7.99212	11.09538	14.21517	17.34274	20.47457	23.60897	26.74506
0.95	2.04597	4.92303	7.98505	11.09021	14.21110	17.33939	20.47172	23.60651	26.74288
1.00	2.02876	4.91318	7.97867	11.08554	14.20744	17.33638	20.46917	23.60428	26.74092

λ^{-1}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
-1.00	2.02876	4.91318	7.97867	11.08554	14.20744	17.33638	20.46917	23.60428	26.74092
-0.95	2.01194	4.90375	7.97258	11.08110	14.20395	17.33351	20.46673	23.60217	26.73905
-0.90	1.99465	4.89425	7.96648	11.07665	14.20045	17.33064	20.46430	23.60006	26.7371
-0.85	1.97687	4.88468	7.96036	11.07219	14.19697	17.32777	20.46187	23.59795	26.7353
-0.80	1.95857	4.87504	7.95422	11.06773	14.19347	17.32490	20.45943	23.59584	26.7334
-0.75	1.93974	4.86534	7.94807	11.06326	14.18997	17.32203	20.45700	23.59372	26.7315
-0.70	1.92035	4.85557	7.94189	11.05879	14.18647	17.31915	20.45456	23.59161	26.7297
-0.65	1.90036	4.84573	7.93571	11.05431	14.18296	17.31628	20.45212	23.58949	26.7278
-0.60	1.87976	4.83583	7.92950	11.04982	14.17946	17.31340	20.44968	23.58738	26.7259
-0.55	1.85852	4.82587	7.92329	11.04533	14.17594	17.31052	20.44724	23.58526	26.7241
-0.50	1.83660	4.81584	7.91705	11.04083	14.17243	17.30764	20.44480	23.58314	26.7222
-0.45	1.81396	4.80575	7.91080	11.03633	14.16892	17.30476	20.44236	23.58102	26.7203
-0.40	1.79058	4.79561	7.90454	11.03182	14.16540	17.30187	20.43992	23.57891	26.7185
-0.35	1.76641	4.78540	7.89827	11.02730	14.16188	17.29899	20.43748	23.57679	26.7166
-0.30	1.74140	4.77513	7.89198	11.02278	14.15835	17.29611	20.43503	23.57467	26.7147
-0.25	1.71551	4.76481	7.88567	11.01826	14.15483	17.29321	20.43259	23.57255	26.7129
-0.20	1.68868	4.75443	7.87936	11.01373	14.15130	17.29033	20.43014	23.57043	26.7110
-0.15	1.66087	4.74400	7.87303	11.00920	14.14777	17.28744	20.42769	23.56831	26.7091
-0.10	1.63199	4.73351	7.86669	11.00466	14.14424	17.28454	20.42525	23.56619	26.7072
-0.05	1.60200	4.72298	7.86034	11.00012	14.14070	17.28165	20.42280	23.56407	26.7054
0.00	1.57080	4.71239	7.85398	10.99557	14.13717	17.27875	20.42035	23.56194	26.7035
0.05	1.53830	4.70176	7.84761	10.99102	14.13363	17.27586	20.41790	23.55982	26.7016
0.10	1.50442	4.69108	7.84123	10.98647	14.13009	17.27297	20.41545	23.55770	26.6997
0.15	1.46904	4.68035	7.83484	10.98192	14.12655	17.27007	20.41300	23.55558	26.6979
0.20	1.43203	4.66958	7.82844	10.97736	14.12301	17.26718	20.41055	23.55345	26.6960
0.25	1.39325	4.65878	7.82203	10.97279	14.11946	17.26428	20.40810	23.55133	26.6941
0.30	1.35252	4.64793	7.81562	10.96823	14.11592	17.26138	20.40565	23.54921	26.6923
0.35	1.30965	4.63705	7.80919	10.96366	14.11237	17.25848	20.40320	23.54708	26.6904
0.40	1.26440	4.62614	7.80276	10.95909	14.10882	17.25559	20.40075	23.54496	26.6885
0.45	1.21649	4.61519	7.79633	10.95452	14.10527	17.25268	20.39829	23.54283	26.6866
0.50	1.16556	4.60422	7.78988	10.94994	14.10172	17.24978	20.39584	23.54071	26.6848
0.55	1.11118	4.59321	7.78344	10.94537	14.09817	17.24688	20.39339	23.53858	26.6829
0.60	1.05279	4.58219	7.77698	10.94079	14.09462	17.24398	20.39094	23.53646	26.6810
0.65	0.98966	4.57114	7.77053	10.93621	14.09107	17.24108	20.38848	23.53433	26.6791
0.70	0.92079	4.56007	7.76407	10.93163	14.08752	17.23817	20.38603	23.53221	26.6773
0.75	0.84473	4.54899	7.75760	10.92704	14.08396	17.23527	20.38357	23.53008	26.6754
0.80	0.75931	4.53789	7.75114	10.92246	14.08041	17.23237	20.38112	23.52796	26.6735
0.85	0.66086	4.52678	7.74467	10.91788	14.07686	17.22946	20.37867	23.52583	26.6716
0.90	0.54228	4.51566	7.73820	10.91329	14.07330	17.22655	20.37621	23.52370	26.6698
0.95	0.38537	4.50454	7.73172	10.90871	14.06975	17.22366	20.37376	23.52158	26.6679
1.00	0.00000	4.49341	7.72525	10.90412	14.06619	17.22075	20.37130	23.51945	26.66

Raízes da Equação ctg(x_n) = λx_n
(Abramovitz & Stegun p. 225)



λ	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0.00	1.57080	4.71239	7.85398	10.99557	14.13717	17.27876	20.42035	23.56194	26.70354
0.05	1.49613	4.49148	7.49541	10.51167	13.54198	16.58639	19.64394	22.71311	25.79232
0.10	1.42887	4.30580	7.22811	10.20026	13.21418	16.25936	19.32703	22.41085	25.50638
0.15	1.36835	4.15504	7.04126	10.01222	13.03901	16.10053	19.18401	22.28187	25.38952
0.20	1.31384	4.03357	6.90960	9.89275	12.93522	16.01066	19.10552	22.21256	25.32765
0.25	1.26459	3.93516	6.81401	9.81188	12.86775	15.95363	19.05645	22.16965	25.28961
0.30	1.21995	3.85460	6.74233	9.75407	12.82073	15.91443	19.02302	22.14058	25.26392
0.35	1.17933	3.78784	6.68698	9.71092	12.78621	15.88591	18.99882	22.11960	25.24544
0.40	1.14223	3.73184	6.64312	9.67758	12.75985	15.86426	18.98052	22.10377	25.23150
0.45	1.10820	3.68433	6.60761	9.65109	12.73907	15.84728	18.96619	22.09140	25.22062
0.50	1.07687	3.64360	6.57833	9.62956	12.72230	15.83361	18.95468	22.08147	25.21190
0.55	1.04794	3.60834	6.55380	9.61173	12.70847	15.82237	18.94523	22.07333	25.20475
0.60	1.02111	3.57756	6.53297	9.59673	12.69689	15.81297	18.93734	22.06653	25.19878
0.65	0.99617	3.55048	6.51508	9.58394	12.68704	15.80500	18.93065	22.06077	25.19373
0.70	0.97291	3.52649	6.49954	9.57292	12.67857	15.79814	18.92490	22.05583	25.18939
0.75	0.95116	3.50509	6.48593	9.56331	12.67121	15.79219	18.91991	22.05154	25.18563
0.80	0.93076	3.48590	6.47392	9.55486	12.66475	15.78698	18.91554	22.04778	25.18234
0.85	0.91158	3.46859	6.46324	9.54738	12.65904	15.78237	18.91168	22.04447	25.17943
0.90	0.89352	3.45292	6.45368	9.54072	12.65395	15.77827	18.90825	22.04151	25.17684
0.95	0.87647	3.43865	6.44508	9.53473	12.64939	15.77459	18.90518	22.038[illegible]	25.17453
1.00	0.86033	3.42562	6.43730	9.52933	12.64529	15.77128	18.90241	22.03650	25.17245

λ^{-1}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
1.00	0.86033	3.42562	6.43730	9.52933	12.64529	15.77128	18.90241	22.03650	25.17245
0.95	0.84426	3.41306	6.42987	9.52419	12.64138	15.76814	18.89978	22.03424	25.17047
0.90	0.82740	3.40034	6.42241	9.51904	12.63747	15.76499	18.89715	22.03197	25.16848
0.85	0.80968	3.38744	6.41492	9.51388	12.63355	15.76184	18.89451	22.02971	25.16650
0.80	0.79103	3.37438	6.40740	9.50871	12.62963	15.75868	18.89188	22.02745	25.16452
0.75	0.77136	3.36113	6.39984	9.50353	12.62570	15.75553	18.88924	22.02519	25.16254
0.70	0.75056	3.34772	6.39226	9.49834	12.62177	15.75237	18.88660	22.02292	25.16055
0.65	0.72851	3.33413	6.38464	9.49314	12.61784	15.74921	18.88396	22.02066	25.1585
0.60	0.70507	3.32037	6.37700	9.48793	12.61390	15.74605	18.88132	22.01839	25.1565
0.55	0.68006	3.30643	6.36932	9.48271	12.60996	15.74288	18.87868	22.01612	25.1546
0.50	0.65327	3.29231	6.36162	9.47749	12.60601	15.73972	18.87604	22.01386	25.1526
0.45	0.62444	3.27802	6.35389	9.47225	12.60206	15.73655	18.87339	22.01159	25.1506
0.40	0.59324	3.26355	6.34613	9.46700	12.59811	15.73338	18.87075	22.00932	25.1486
0.35	0.55922	3.24891	6.33835	9.46175	12.59415	15.73021	18.86810	22.00705	25.1466
0.30	0.52179	3.23409	6.33054	9.45649	12.59019	15.72704	18.86546	22.00478	25.1446
0.25	0.48009	3.21910	6.32270	9.45122	12.58623	15.72386	18.86281	22.00251	25.1426
0.20	0.43284	3.20393	6.31485	9.44595	12.58226	15.72068	18.86016	22.00024	25.1407
0.15	0.37788	3.18860	6.30696	9.44067	12.57829	15.71751	18.85751	21.99797	25.1387
0.10	0.31105	3.17310	6.29906	9.43538	12.57432	15.71433	18.85486	21.99569	25.1367
0.05	0.22176	3.15743	6.29113	9.43008	12.57035	15.71114	18.85221	21.99342	25.1347
0.00	0.00000	3.14159	6.28319	9.42478	12.56637	15.70796	18.84956	21.99115	25.1327

(3) $$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}, \quad 0 < x < a,\ t > 0 \\ T(x,0) = x \\ \left.\dfrac{\partial T}{\partial x}\right|_{0,t} = \left.\dfrac{\partial T}{\partial x}\right|_{a,t} = 0 \end{cases}$$

R.: $T(x,t) = \dfrac{a}{2} + \dfrac{2a}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos(n\pi)-1}{n^2}\cos(\lambda_n x)\exp\left[-\alpha\lambda_n{}^2 t\right]$

- $\lambda_n = \dfrac{n\pi}{a}$ são as raízes de $\operatorname{sen}(\lambda_n a) = 0$.

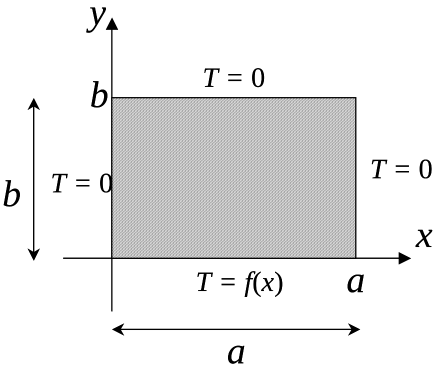
(4) $$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}, \quad 0 < x < a,\ t > 0 \\ T(x,0) = f(x) \\ T(0,t) = A \\ T(a,t) = B \end{cases}$$

Sugestão: $T(x,t) = g(x) + X(x)T(t)$

R.: $T(x,t) = \dfrac{B-A}{a}x + A + \dfrac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a \left(f(x) - g(x)\right)\operatorname{sen}(\lambda_n x)\,dx\right]\operatorname{sen}(\lambda_n x)\exp\left[-\alpha\lambda_n{}^2 t\right]$

- $\lambda_n = \dfrac{n\pi}{a}$ são as raízes de $\operatorname{sen}(\lambda_n a) = 0$.

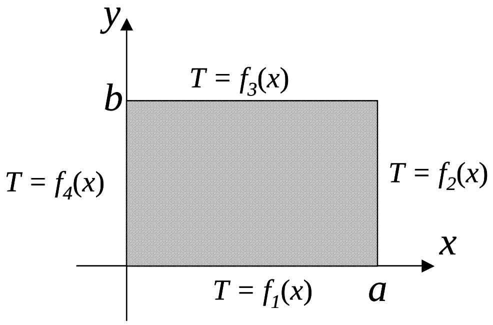
(5) $$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b \\ T(0,y) = T(a,y) = 0 \\ T(x,b) = 0, \quad T(x,0) = f(x) \end{cases}$$



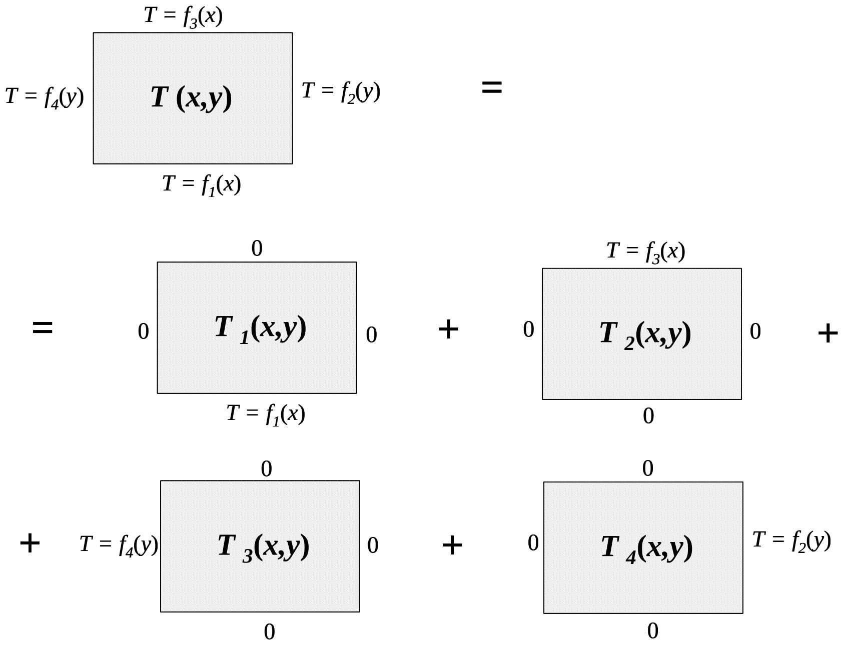
R.: $T(x,t) = -\frac{2}{a}\sum_{n=1}^{\infty}\left[\int_0^a f(x)\operatorname{sen}(\lambda_n x)\,dx\right]\frac{\operatorname{senh}(\lambda_n y) - \tanh(\lambda_n b)\cdot\cosh(\lambda_n y)}{\tanh(\lambda_n b)}\operatorname{sen}(\lambda_n x)$

- $\lambda_n = \frac{n\pi}{a}$

(6)
$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b \\ T(0,y) = f_4(y), \quad T(a,y) = f_2(y) \\ T(x,b) = f_3(x), \quad T(x,0) = f_1(x) \end{cases}$$



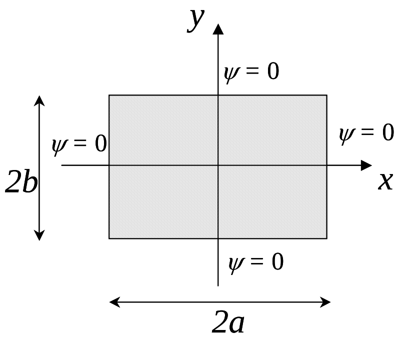
Sugestão: pelo Princípio da Superposição,



R.: $T(x,y) = \underset{\substack{\text{S-L em } x \\ \text{(prob. 5)}}}{T_1(x,y)} + \underset{\text{S-L em } x}{T_2(x,y)} + \underset{\text{S-L em } y}{T_3(x,y)} + \underset{\text{S-L em } y}{T_4(x,y)}$

(S-L: problema de Sturm-Liouville)

(7) $$\begin{cases} \dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}+2=0, \quad -a<x<a\,,\ -b<y<b \quad \text{(Equação de Poisson)} \\ \psi=0 \ \text{ ao longo do perímetro} \end{cases}$$

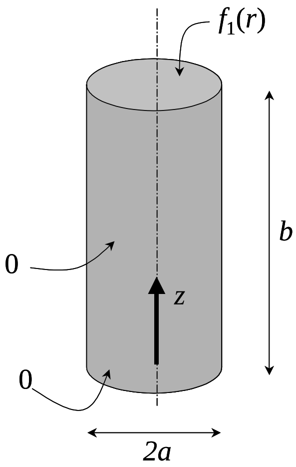


Sugestão: $\psi(x,y)= f(x)+g(x,y)$

R.: $$\psi(x,y)=a^2-x^2+\frac{2}{a}\sum_{n=0}^{\infty}\left[\int_0^a\left(x^2-a^2\right)\cos(\lambda_n x)\,dx\right]\frac{\cosh(\lambda_n y)}{\cosh(\lambda_n b)}\cos(\lambda_n x)$$

$\lambda_n=\dfrac{2n+1}{2a}\pi\ ,\quad n=0{,}1{,}2{,}...$ são as raízes de $\cos(\lambda_n a)=0$.

(8) $$\begin{cases} \dfrac{\partial^2 T}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial T}{\partial r}+\dfrac{\partial^2 T}{\partial z^2}=0, \quad 0<r<a\,,\ 0<z<b \\ \left.\dfrac{\partial T}{\partial r}\right|_{0,z}=T(a,z)=0 \\ T(r,0)=0, \quad T(r,b)=f_1(r) \end{cases}$$



R.: $$T(r,z)=\sum_{n=1}^{\infty}\frac{\int_0^a rf_1(r)J_0(\lambda_n r)\,dr}{\int_0^a rJ_0{}^2(\lambda_n r)\,dr}\cdot\frac{\operatorname{senh}(\lambda_n z)}{\operatorname{senh}(\lambda_n b)}\cdot J_0(\lambda_n r)$$

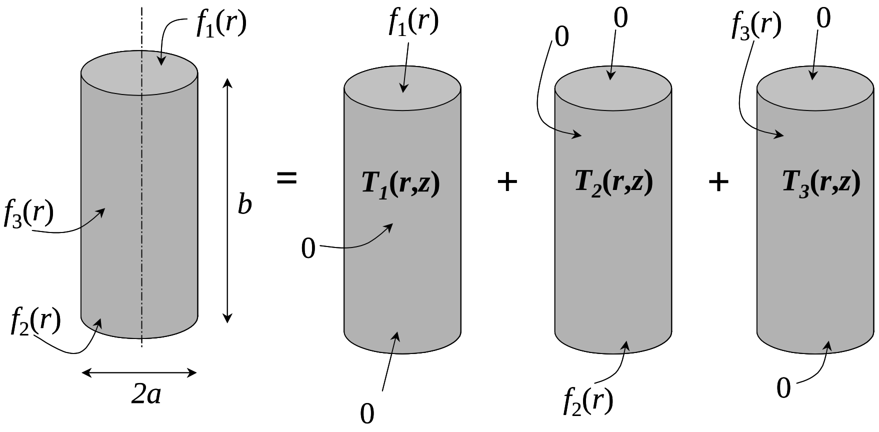
- λ_n são as raízes de $J_0(\lambda_n a)=0$:

$\lambda_1 a$	$\lambda_2 a$	$\lambda_3 a$	$\lambda_4 a$	$\lambda_5 a$	$\lambda_6 a$
2,405	5,520	8,654	11,79	14,93	18,07

$$\underbrace{\qquad}_{3{,}12}\quad\underbrace{\qquad}_{3{,}13}\quad\underbrace{\qquad}_{3{,}14}\quad\underbrace{\qquad}_{3{,}14}\quad\underbrace{\qquad}_{3{,}14}\quad \to \pi$$

(ver Spiegel & Liu)

(9) $\dfrac{\partial^2 T}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial T}{\partial r}+\dfrac{\partial^2 T}{\partial z^2}=0, \quad 0<r<a,\ 0<z<b$

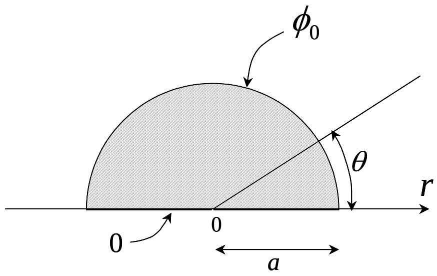


R.: $T(r,z)=\underset{\text{S-L em } r}{T_1(r,z)}+\underset{\text{S-L em } r}{T_2(r,z)}+\underset{\text{S-L em } z}{T_3(r,z)}$

$$\begin{cases} T_1(r,z)=\displaystyle\sum_{n=1}^{\infty} A_n \frac{\operatorname{senh}(\lambda_n z)}{\operatorname{senh}(\lambda_n b)}\cdot J_0(\lambda_n r) \\ J_0(\lambda_n a)=0 \\ A_n=\dfrac{2}{a^2 J_1{}^2(\lambda_n a)}\displaystyle\int_0^a r f_1(r) J_0(\lambda_n r)\,dr \end{cases} \qquad \begin{cases} T_2(r,z)=\displaystyle\sum_{n=1}^{\infty} B_n \frac{\operatorname{senh}[(b-z)\lambda_n]}{\operatorname{senh}(\lambda_n b)}\cdot J_0(\lambda_n r) \\ J_0(\lambda_n a)=0 \\ B_n=\dfrac{2}{a^2 J_1{}^2(\lambda_n a)}\displaystyle\int_0^a r f_2(r) J_0(\lambda_n r)\,dr \end{cases}$$

$$\begin{cases} T_3(r,z)=\displaystyle\sum_{n=1}^{\infty} C_n \operatorname{sen}\left(\frac{n\pi}{b}z\right)\frac{I_0\left(\frac{n\pi}{b}r\right)}{I_0\left(\frac{n\pi}{b}a\right)} \\ B_n=\dfrac{2}{b}\displaystyle\int_0^b f_3(z)\operatorname{sen}\left(\frac{n\pi}{b}z\right)dz \end{cases}$$

(10) $$\begin{cases} \dfrac{\partial^2\phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}=0, \quad r\le a,\; 0\le\theta\le\pi \\ \phi(a,\theta)=\phi_0 \text{ , constante} \\ \phi(r,0)=0 \\ \phi(r,\pi)=0 \end{cases}$$



R.: $\phi(r,\theta)=\sum_{n=1}^{\infty}E_n r^n \operatorname{sen}(n\theta)$ $\qquad E_n=\dfrac{\int_0^{\pi}\phi_0 \operatorname{sen}(n\theta)d\theta}{a^n\int_0^{\pi}\operatorname{sen}^2(n\theta)d\theta}$

(11) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial u}{\partial y}=u\,, \quad 0<x<\pi\,, \quad y>1 \\ u(0,y)=0 \\ u(\pi,y)=0 \\ u(x,1)=x \end{cases}$$

R.: $u(x,y)=-2\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}e^{(1+n^2)(1-y)}\sin(nx)$

(12) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{\alpha}\dfrac{\partial^2 u}{\partial t^2}\,, \quad 0<x<l\,, \quad t>0 \\ u(x,0)=\varphi(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{x,0}=0 \\ u(0,t)=u(l,t)=0 \end{cases}$$

R.: $u(x,t)=\dfrac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l \varphi(x)\sin\left(\dfrac{n\pi x}{l}\right)dx\right]\sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi a}{l}t\right)$

(13) $$\begin{cases} \dfrac{\partial^2\theta}{\partial\beta^2} + \dfrac{1}{\beta}\dfrac{\partial\theta}{\partial\beta} = \dfrac{\partial\theta}{\partial\tau}, \quad 0 < \beta < 1, \quad \tau > 0 \\ \theta(\beta, 0) = 1 \\ \theta(0, \tau) \text{ é finito} \\ \left.\dfrac{\partial\theta}{\partial\beta}\right|_{1,\tau} + \Omega\theta(1,\tau) = 0, \quad \Omega \text{ constante} \end{cases}$$

R.: $\theta(\beta,\tau) = \sum_{n=1}^{\infty} D_n J_0(\lambda_n\beta)\exp(-\lambda_n\beta)$, $\qquad D_n = \dfrac{\int_0^1 \beta J_0(\lambda_n\beta)\, d\beta}{\int_0^1 \beta J_0{}^2(\lambda_n\beta)\, d\beta}$

Para achar os λ_n: $\lambda_n J_1(\lambda_n) + \Omega J_0(\lambda_n) = 0$

Ω	λ_1	λ_2	λ_3	λ_4
0	0,0000	3,8317	7,0156	10,1735
1	1,2558	4,0795	7,1558	10,2710
∞	3,1153	5,5201	8,6537	11,7942

3,1153 3,1336 3,1405 $\to \pi$

(Abramovitz e Stegun, 1965)

(14) $$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0, \quad r > 1, \quad -\pi < \theta < \pi \\ u(1,\theta) = \theta \\ u(r,-\pi) = u(r,\pi) = 0 \end{cases}$$

R.: $u\left(r,\theta\right)=2\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n}r^{-n}\,\sin\left(n\theta\right)$

(15) $$\begin{cases}\alpha\dfrac{\partial^2 v}{\partial x^2}=\beta\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial t}, \quad 0<x<a, \quad t>0\\ v\left(x,0\right)=0\\ v\left(0,t\right)=0, \quad v\left(a,t\right)=V \text{ (constante)}\end{cases}$$

Sugestão: fazer $v\left(x,t\right)=f\left(x\right)+\Omega\left(x,t\right)$

R.: $v\left(x,t\right)=f\left(x\right)+\sum_{n=1}^{\infty}D_n^{*}\exp\left(\frac{-\beta}{2\alpha}x\right)\sin\left(\frac{n\pi}{a}x\right)\exp\left(-\lambda_n{}^2 t\right)$

$$f\left(x\right)=\frac{\exp\left(\dfrac{\beta}{\alpha}x\right)-1}{\exp\left(\dfrac{\beta}{\alpha}a\right)-1}V$$

$$D_n^{*}=-\frac{\int_0^a f\left(x\right)\dfrac{1}{\alpha}\exp\left(-\dfrac{\beta}{\alpha}x\right)\exp\left(\dfrac{\beta}{2\alpha}x\right)\sin\left(\dfrac{n\pi}{a}x\right)dx}{\int_0^a \dfrac{1}{\alpha}\exp\left(-\dfrac{\beta}{\alpha}x\right)\left[\exp\left(\dfrac{\beta}{2\alpha}x\right)\sin\left(\dfrac{n\pi}{a}x\right)\right]^2 dx}$$

7 – Transformadas Integrais

$$\boxed{\overline{f}(p) = \int_a^b f(x)K(p,x)\,dx}$$

Transformada	$\overline{f}(p)$	$f(x)$	Transformada de $f'(x)$ e $f''(x)$
Complexa de Fourier	$E\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)e^{-ipx}\,dx$	$E^{-1}\left[\overline{f}(p)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(p)e^{ipx}\,dp$	$E\left[f''(x)\right] = -p^2E\left[f(x)\right]$ $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} f'(x) = 0$
Cosseno de Fourier	$C\left[f(x)\right] = \int_0^{\infty} f(x)\cos(px)\,dx$	$C^{-1}\left[\overline{f}(p)\right] = \frac{2}{\pi}\int_0^{\infty} \overline{f}(p)\cos(px)\,dp$	$C\left[f''(x)\right] = -p^2C\left[f(x)\right] - f'(0)$ $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$
Seno de Fourier	$S\left[f(x)\right] = \int_0^{\infty} f(x)\sin(px)\,dx$	$S^{-1}\left[\overline{f}(p)\right] = \frac{2}{\pi}\int_0^{\infty} \overline{f}(p)\sin(px)\,dp$	$S\left[f''(x)\right] = -p^2S\left[f(x)\right] - pf(0)$ $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$
Laplace	$L\left[f(x)\right] = \int_0^{\infty} f(x)e^{-px}\,dx$	$L^{-1}\left[\overline{f}(p)\right] = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{a-i\omega}^{a+i\omega} e^{st}f(s)\,ds$	$L\left[f'(x)\right] = pL\left[f(x)\right] - f(0)$ $L\left[f''(x)\right] = p^2L\left[f(x)\right] - pf(0) - f'(0)$

Transformada	**Ordem da Derivada**	**Domínio da Variável**	**Condições Limites “Naturais”**
Complexa de Fourier	2	$(-\infty,\infty)$	$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty} f'(x)=0$
Cosseno de Fourier	2	$[0,\infty)$	Em $f'(0)$ $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} f'(x)=0$
Seno de Fourier	2	$[0,\infty)$	Em $f(0)$ $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} f'(x)=0$
Laplace	1	$[0,\infty)$	Em $f(0)$
Laplace	2	$[0,\infty)$	Em $f(0)$ e $f'(0)$

Transformada de Laplace: Teoremas

$$\boxed{L\left[f\left(t\right)\right]=\int_0^{\infty} f\left(t\right)e^{-st}\,dt=\overline{f}\left(s\right)}$$

$L\left[f\left(t\right)\right]$	$f\left(t\right)$
$aL\left[f_1\left(t\right)\right]+bL\left[f_2\left(t\right)\right]$	$af_1\left(t\right)+bf_2\left(t\right)$
$\frac{1}{k}\overline{f}\left(\frac{s}{k}\right)$	$f\left(kt\right)$
$e^{-sk}\overline{f}\left(s\right)$	$u\left(t-k\right)f\left(t-k\right)$
$\overline{f}\left(s-k\right)$	$e^{kt}f\left(t\right)$
$-\frac{d\overline{f}\left(s\right)}{ds}$	$tf\left(t\right)$
$L\left[f_1\left(t\right)\right]\cdot L\left[f_2\left(t\right)\right]$	$f_1\left(t\right)\cdot f_2\left(t\right)=\int_0^t f_1\left(\eta\right)f_2\left(t-\eta\right)d\eta$ $=\int_0^t f_1\left(t-\eta\right)f_2\left(\eta\right)d\eta$
$\overline{f}\left(s\right)$ é uma função analítica exceto para um número finito de pólos $s=a_i$ (ou infinito porém enumerável)	$f\left(t\right)=L^{-1}\left[\overline{f}\left(s\right)\right]=\sum_i \underset{s=a_i}{\mathrm{Res}}\left[e^{st}\,\overline{f}\left(s\right)\right]$ $s=a_i$ é um pólo simples: $\underset{s=a_i}{\mathrm{Res}}\left[e^{st}\overline{f}\left(s\right)\right]=\lim_{s\to a_i}\left\{\left(s-a_i\right)e^{st}\overline{f}\left(s\right)\right\}$ $s=a_i$ é um pólo duplo: $\underset{s=a_i}{\mathrm{Res}}\left[e^{st}\overline{f}\left(s\right)\right]=\lim_{s\to a_i}\left\{\frac{d}{ds}\left(s-a_i\right)e^{2st}\overline{f}\left(s\right)\right\}$

Antes de empreender a longa viagem de volta, verifique se:

$$\lim_{s\to\infty} sL\left[f\left(t\right)\right]=\lim_{t\to 0} f\left(t\right),\quad \lim_{s\to 0} sL\left[f\left(t\right)\right]=\lim_{t\to\infty} f\left(t\right)$$

Transformada de Laplace de algumas funções

Referência: Carslaw & Jaeger, “Operation Methods in Applied Mathematics”, Dover, N. York

$L[f(t)]$	$f(t)$
$s^{-1-\frac{n}{2}} \cdot e^{-a\sqrt{s}}$	$(4t)^{\frac{n}{2}} \cdot \boldsymbol{i}^{n} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$ $n = 0,1,2,\ldots$
$\frac{e^{-a\sqrt{s}}}{b+\sqrt{s}}$	$\frac{e^{-\frac{a^2}{4t}}}{\sqrt{\pi t}} - b e^{ab+b^2 t} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} + b\sqrt{t}\right)$
$\frac{e^{-a\sqrt{s}}}{s\left(b+\sqrt{s}\right)}$	$\frac{\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)}{b} - \frac{e^{ab+b^2 t}}{b} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} + b\sqrt{t}\right)$
$\frac{e^{-a\sqrt{s}}}{s-\omega}$	$\frac{e^{\omega t}}{2}\left\{e^{-a\sqrt{\omega}} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{\omega t}\right) + e^{a\sqrt{\omega}} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{\omega t}\right)\right\}$
$K_0\left(a\sqrt{s}\right)$	$\frac{e^{-\frac{a^2}{4t}}}{2t}$
$K_0\left(as\right)$	$\begin{cases} 0 & \text{para } 0 < t < a \\ \frac{1}{\sqrt{t^2 - a^2}} & \text{para } t > a \end{cases}$
$K_0\left(b\sqrt{s^2 + a^2}\right)$	$\begin{cases} 0 & \text{para } 0 < t < b \\ \frac{\cos\left(a\sqrt{t^2 - b^2}\right)}{\sqrt{t^2 - b^2}} & \text{para } t > b \end{cases}$
$\pi\, e^{-as}\, I_0\left(as\right)$	$\begin{cases} 0 & \text{para } t > 2a \\ \frac{1}{\sqrt{t\left(2a - t\right)}} & \text{para } 0 < t < 2a \end{cases}$
$e^{-bs} - e^{-b\sqrt{s^2 + a^2}}$	$\begin{cases} 0 & \text{para } 0 < t < b \\ \frac{ab\, J_1\left(a\sqrt{t^2 - b^2}\right)}{\sqrt{t^2 - b^2}} & \text{para } t > b \end{cases}$
$\frac{\left(s - \sqrt{s^2 - a^2}\right)^n}{n\, a^n}$	$\frac{I_n\left(at\right)}{t}$ para $n > 0$

$\dfrac{a^n\, e^{\frac{a}{s}}}{s^{n+1}}$	$(at)^{n/2}\, I_n\left(2\sqrt{at}\right)$ para $n > -1$
$\dfrac{e^{-b\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}$	$\begin{cases} 0 \quad \mathit{para} \quad 0 < t < b \\ I_0\left(a\sqrt{t^2-b^2}\right) \text{ para } t > b \end{cases}$
$e^{-b\sqrt{s^2-a^2}} - e^{-bs}$	$\begin{cases} 0 \quad \mathit{para} \quad 0 < t < b \\ \dfrac{ab\, I_1\left(a\sqrt{t^2-b^2}\right)}{\sqrt{t^2-b^2}} \text{ para } t > b \end{cases}$
$\dfrac{\left(s-\sqrt{s^2-a^2}\right)^2}{a^n\sqrt{s^2-a^2}} \cdot e^{-b\sqrt{s^2-a^2}}$	$\begin{cases} 0 \quad \mathit{para} \quad 0 < t < b \\ \left(\dfrac{t-b}{t+b}\right)^{n/2} \cdot I_n\left(a\sqrt{t^2-b^2}\right) \text{ para } t > b \text{ e } n > -1 \end{cases}$

onde:

$$\boldsymbol{i}^n\, erfc(\eta) = \int_\eta^\infty \boldsymbol{i}^{n-1}\, \mathrm{erfc}(\xi)\, d\xi \text{ para } n = 1,2,\ldots\text{, com } \boldsymbol{i}^0\, \mathrm{erfc}(\eta) = \mathrm{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty e^{-\xi^2}\, d\xi$$

Série nº 9 – Transformada de Laplace

Resolver:

(1) $$\begin{cases} y''+3y'+2y=u(t-1) \\ y(0)=0, \quad y'(0)=1 \end{cases}$$

R.: $y(t)=e^{-t}-e^{-2t}+\left[-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}+\frac{1}{2}\right]u(t-1)$

(2) $$\begin{cases} y'+2y+6\int_0^t z(\tau)\,d\tau=-2, \quad t>0 \\ y'+z'+z=0 \\ y(0)=-5, \quad z(0)=6 \end{cases}$$

R.: $y(t)=2-4e^{t}-3e^{-4t}, \qquad z(t)=$..................

(3) $$\begin{cases} \dfrac{\partial\Omega}{\partial t}=\nu\,\dfrac{\partial^2\Omega}{\partial y^2}, \quad y>0\,,t>0 \\ \Omega(0,t)=1 \\ \Omega(y,0)=\lim\limits_{y\to\infty}\Omega(y,t)=0 \end{cases}$$

R.: $\Omega(y,t)=\operatorname{erfc}\dfrac{y}{2\sqrt{\nu t}}$

(4) Resolver o sistema para $V=V(x,t)$:

$$\begin{cases} RI = -\dfrac{\partial V}{\partial x}, \quad x > 0, \quad t > 0 \\ C\dfrac{\partial V}{\partial t} = -\dfrac{\partial I}{\partial x} \\ t = 0, \quad I(x,t) = V(x,t) = 0 \\ x = 0, \quad V(x,t) = V_0, \quad \text{constante} \end{cases}$$

$\lim_{x\to\infty} V(x,t)$ é finito, R e C são constantes.

R.: $V(x,t) = V_0 \operatorname{erfc}\left[\dfrac{(CR)^{1/2}\, x}{2\sqrt{t}}\right]$

(5) $$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < a, \quad t > 0 \\ u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial t}\right|_{x,0} = 0 \\ u(0,t) = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{a,t} = k, \quad \text{constante} \end{cases}$$

R.: $u(x,t) = ck\left[\dfrac{x}{c} + \dfrac{8a}{c\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)^2}\sin\dfrac{(2n-1)\pi x}{2a}\cos\dfrac{(2n-1)\pi ct}{2a}\right]$

(6) $$\begin{cases} \dfrac{\partial V}{\partial x} + 2x\dfrac{\partial V}{\partial t} = 2x, \quad x > 0, \quad t > 0 \\ V(x,0) = V(0,t) = 1 \end{cases}$$

R.: $V(x,t) = \begin{cases} 1 + t, \quad t < x^2 \\ 1 + x^2, \quad t \geq x^2 \end{cases}$

(7) $$\begin{cases} \dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} = \dfrac{1}{k}\dfrac{\partial U}{\partial t} \\ U(r,0) = 0, \quad 0 < x < a \\ U(a,t) = U_0, \quad t > 0 \end{cases}$$

R.: $$U(r,t) = U_0 \left\{ 1 - 2\sum_{n=1}^{\infty} \frac{J_0\left(\dfrac{\lambda_n r}{a}\right)}{\lambda_n J_1(\lambda_n)} \exp\left[-k\left(\frac{\lambda_n}{a}\right)^2 t \right] \right\}$$

λ_n são as raízes positivas de $J_0(\lambda) = 0$.

(8) $F'(t) + k^2 \int_0^t F(x)\cosh\left[k(t-x)\right] dx = 0, \quad F(0) = C$

R.: $F(t) = C\left(1 - \dfrac{k^2 t^2}{2}\right)$

(9) $$\begin{cases} \dfrac{\partial T}{\partial x} + x\dfrac{\partial T}{\partial t} = 0, \quad x > 0, \quad t > 0 \\ T(x,0) = 0, \quad T(0,t) = t \end{cases}$$

R.: $T(x,t) = \left(t - \dfrac{x^2}{2}\right) u\left(t - \dfrac{x^2}{2}\right)$

(10) $$\begin{cases} \left(1+k\theta\right)c_{n+1}-c_n = -\theta\dfrac{\partial c_{n+1}}{\partial t} \\ c_n\left(0\right)=0 \\ c_0\left(t\right)=A, \quad \text{constante} \end{cases}$$

k e θ são constantes

R.: $c_n\left(t\right)=\dfrac{A}{\theta^n\,\Gamma\left(n\right)}\displaystyle\int_0^t u^{n-1}\,e^{-\left(k+\frac{1}{\theta}\right)u}\,du$

(11) $$\begin{cases} \dfrac{\partial U}{\partial t}=k\dfrac{\partial U}{\partial x^2}-hU\,, \quad x>0\,,\ t>0 \\ U\left(x,0\right)=0 \\ U\left(0,t\right)=F_0\,, \quad \lim\limits_{x\to\infty} U\left(x,t\right)=0 \end{cases}$$

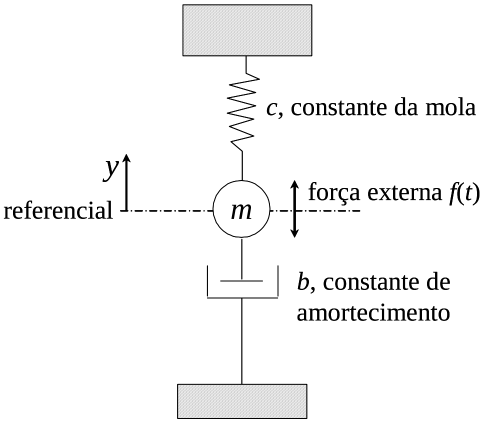
R.: $U\left(x,t\right)=F_0\,\dfrac{x}{\sqrt{k}}\displaystyle\int_0^t \dfrac{1}{2\sqrt{\pi u^3}}\exp\left[-\left(hu+\dfrac{x^2}{4ku}\right)\right]du$

(12) $$\begin{cases} x\dfrac{\partial T}{\partial x}+\dfrac{\partial T}{\partial t}+T = xF\left(t\right), \quad x>0\,, \quad t>0 \\ T\left(x,0\right)=T\left(0,t\right)=0 \end{cases}$$

R.: $T\left(x,t\right)=xe^{-2t}\displaystyle\int_0^t e^{2\eta}\,\overline{F}\left(\eta\right)d\eta$

(13) Sistema mecânico com um grau de liberdade

$$\begin{cases} a\dfrac{d^2y}{dt^2}+b\dfrac{dy}{dt}+cy=f(t), \quad a=m/g \\ y(0)=y_0, \quad y'(0)=y'_0, \quad \text{constantes} \end{cases}$$



R.:

$$y(t)=f(t)\cdot h(t)+y_0\psi_1(t)+y'_0\psi_2(t),$$

em que

$h(t)=\dfrac{1}{a}y_H(t), \quad \psi_1(t)=y'_H(t)+\dfrac{b}{a}y_H(t), \quad \psi_2(t)=y_H(t)$, e a função

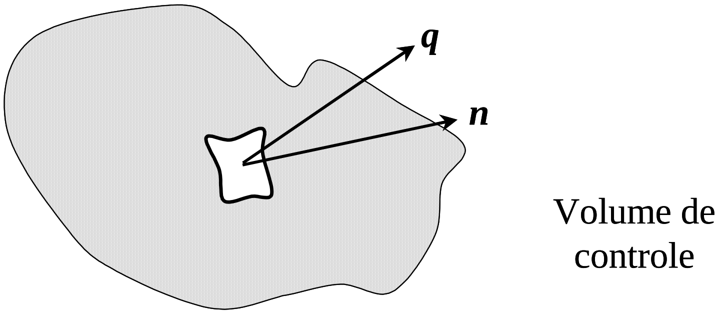
y_H é a solução de $\begin{cases} \dfrac{d^2y_H}{dt^2}+\dfrac{b}{a}\dfrac{dy_H}{dt}+\dfrac{c}{a}y_H=0 \\ y_H(0)=0, \quad y_H'(0)=1 \end{cases}$

(14) $\begin{cases} -\dfrac{\partial T_F}{\partial t}=T_F-T_S \\ \dfrac{\partial T_S}{\partial t}=T_F-T_S \\ T_F(0,t)=T_{Fa}, \quad T_S(z,0)=T_{S0} \end{cases}$

R.: $\dfrac{T_S-T_{S0}}{T_{Fa}-T_{S0}}=\exp(-z)\int_0^t \exp(-u)I_0\left(2\sqrt{zu}\right)du$

$$\frac{T_{Fa}-T_F}{T_{Fa}-T_{S0}}=\exp(-t)\int_0^z \exp(-u)I_0\left(2\sqrt{tu}\right)du$$

8 – Gênese da Equação Diferencial



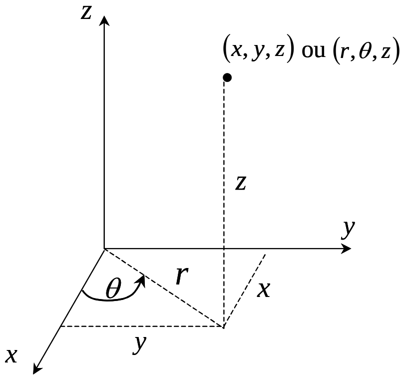
$$\left\{\begin{array}{c}\text{Entra – sai, por}\\ \text{unidade de tempo,}\\ \text{da grandeza}\\ \text{conservativa através}\\ \text{da superfície do VC}\end{array}\right\} + \left\{\begin{array}{c}\text{Geração da}\\ \text{grandeza}\\ \text{conservativa,}\\ \text{por unidade de}\\ \text{tempo, no VC}\end{array}\right\} = \left\{\begin{array}{c}\text{Acumulação no}\\ \text{tempo da grandeza}\\ \text{conservativa}\end{array}\right\}$$

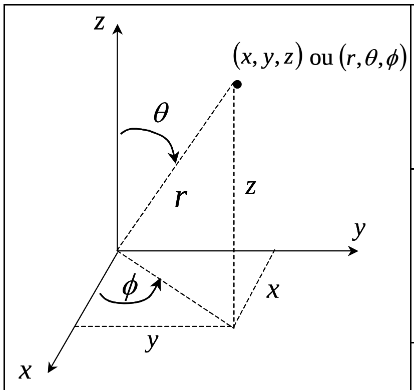
$$-\int_{SC} \boldsymbol{q} \cdot \boldsymbol{n}\, dS + \int_{VC} A\, dV = \int_{VC} \frac{\partial W}{\partial t}\, dV$$

$$\int_{SC} \boldsymbol{q} \cdot \boldsymbol{n}\, dS = \int_{VC} \mathrm{div}(\boldsymbol{q})\, dV \quad \text{(Teorema da Divergência)}$$

$$\boxed{-\,\mathrm{div}(\boldsymbol{q}) + A = \frac{\partial W}{\partial t}}$$

Coordenadas retangulares:	Divergente	$(\nabla \cdot v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
	Laplaciano	$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$
	Gradiente	$[\nabla s]_x = \frac{\partial s}{\partial x}, \ [\nabla s]_y = \frac{\partial s}{\partial y}, \ [\nabla s]_z = \frac{\partial s}{\partial z}$

 Coordenadas cilíndricas	Divergente	$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$
	Laplaciano	$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2}$
	Gradiente	$[\nabla s]_r = \frac{\partial s}{\partial r}, \ [\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta}, \ [\nabla s]_z = \frac{\partial s}{\partial z}$

 Coordenadas esféricas	Divergente	$(\nabla \cdot v) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \operatorname{sen}\theta}\frac{\partial}{\partial \theta}(v_\theta \operatorname{sen}\theta)$ $+ \frac{1}{r \operatorname{sen}\theta}\frac{\partial v_\phi}{\partial \phi}$
	Laplaciano	$(\nabla^2 s) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial s}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial \theta}\left(\operatorname{sen}\theta \frac{\partial s}{\partial \theta}\right)$ $+ \frac{1}{r^2 \operatorname{sen}^2 \theta}\frac{\partial^2 s}{\partial \phi^2}$
	Gradiente	$[\nabla s]_r = \frac{\partial s}{\partial r}, \ [\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta},$ $[\nabla s]_\phi = \frac{1}{r \operatorname{sen}\theta}\frac{\partial s}{\partial \theta}$