Problem Set 1.1:

State the order of the differential equation and verify that the given function is a solution.

1. $y' + 4y = 8x$, $y = ce^{-4x} + 2x - \frac{1}{2}$
2. $y'' + 9y = 0$, $y = A\cos 3x + B\sin 3x$
3. $y' - 0.5y = 1$, $y = ce^{0.5x} - 2$
4. $y''' = 6$, $y = x^3 + ax^2 + bx + c$
5. $y' + y\tan x = 0$, $y = c\cos x$
6. $y'' - 2y' + 2y = 0$, $y = e^x(A\cos x + B\sin x)$

Solve the following differential equations.

7. $y' = e^{-2x}$ 8. $y' = xe^{x^2}$ 9. $y' = -\cos\frac{1}{2}x$ 10. $y''' = 48x$

Verify that the given function is a solution of the corresponding differential equation and determine c so that the resulting particular solution satisfies the given initial condition.

11. $y' + y = 1$, $y = ce^{-x} + 1$, $y = 2.5$ when $x = 0$
12. $y' = 2xy$, $y = ce^{x^2}$, $y = 4$ when $x = 1$
13. $xy' = 2y$, $y = cx^2$, $y = 12$ when $x = 2$
14. $yy' = x$, $y^2 - x^2 = c$, $y(0) = 1$
15. $y' = y\cot x$, $y = c\sin x$, $y(-\pi/2) = 2$
16. $yy' + x = 0$, $x^2 + y^2 = c$, $y(\sqrt{2}) = \sqrt{2}$

Find a first-order differential equation involving both y and y' for which the given function is a solution.

17. $y = x^2$ 18. $y = x^3 - 4$ 19. $y = \tan x$ 20. $x^2 + 9y^2 = 9$

Applications, Modeling

21. **(Falling body)** Experiments show that if a body falls in vacuum due to the action of gravity, then its acceleration is constant (equal to $g = 9.80$ meters/sec^2 = 32.1 ft/sec^2; this is called the *acceleration of gravity*). State this law as a differential equation for $y(t)$, the distance fallen as a function of time t (already mentioned in the text), and solve it to get the familiar law

$$y(t) = \tfrac{1}{2}gt^2.$$

(In practice, this also applies to the free fall in air if we can neglect the air resistance, for instance, if we drop a stone or an iron ball.)

22. **(Falling body, general initial conditions)** If in Prob. 21 the body starts at $t = 0$ from initial position $y = y_0$ with initial velocity $v = v_0$, show that then the solution is

$$y(t) = \tfrac{1}{2}gt^2 + v_0t + y_0.$$

23. **(Airplane takeoff)** An airplane taking off from a landing field has a run of 1.8 kilometers. If the plane starts with speed 5 meters/sec, moves with constant acceleration, and makes the run in 40 sec, with what speed does it take off?
24. In Prob. 23, if you want to reduce the take-off speed to 250 km/hr, to what value can you reduce the constant acceleration, the other data being as before?
25. **(Exponential growth)** We know from the text that $y' = y$ with solution $y(x) = ce^x$ governs the growth of a population if the growth rate $y' = dy/dx$ equals the population $y(x)$ present (x = time). (a) What is the particular solution satisfying $y(0) = 3$? (b) What initial amount $y(0)$ is necessary to get $y = 100$ after $x = 2$ [hours]?
26. **(Exponential growth)** If in a culture of yeast the rate of growth $y'(x)$ is proportional to the population present at time x, say, $y' = ky$, verify that $y(x) = ce^{kx}$. If y doubles in 1 day, how much can be expected after 1 week at the same rate of growth? After 2 weeks?
27. **(Malthus's law)** The law in Prob. 26 (growth rate proportional to the population present) is called *Malthus's law*.[3] For the United States, observed values of $y(t) = y_0e^{kt}$, in millions, are as follows.

t	0	30	60	90	120	150	180	190
Year	1800	1830	1860	1890	1920	1950	1980	1990
Population	5.3	13	31	63	105	150	230	250

Use the first two columns for determining y_0 and k. Then calculate values for 1860, 1890, $\cdots$, 1990 and compare them with the observed values. Comment.

28. **(Exponential decay; atmospheric pressure)** Observations show that the rate of change of the atmospheric pressure y with altitude x is proportional to the pressure. Assuming that the pressure at 6000 meters (about 18,000 ft) is half its value y_0 at sea level, find the formula for the pressure at any height.
29. **(Half-life)** The *half-life* of a radioactive substance is the time in which half of a given amount will disappear. What is the half-life of ${}_{88}Ra^{226}$ (in years) in Example 4?
30. **(Interest rates)** Let $y(x)$ be the investment resulting from a deposit y_0 after x years at an interest rate r. Show that

$$y(x) = y_0[1 + r]^x \quad \text{(interest compounded annually)}$$

$$y(x) = y_0[1 + (r/4)]^{4x} \quad \text{(interest compounded quarterly)}$$

$$y(x) = y_0[1 + (r/365)]^{365x} \quad \text{(interest compounded daily)}.$$

Now recall from calculus that $[1 + (1/n)]^n \to e$ as $n \to \infty$, hence $[1 + (r/n)]^{nx} \to e^{rx}$, which gives

$$y(x) = y_0e^{rx} \quad \text{(interest compounded continuously)}.$$

What differential equation does the last function satisfy? Let $y_0 = \$1000.00$ and $r = 8\%$. Compute $y(1)$ and $y(5)$ from each of the four formulas and confirm that there is not much difference between daily and continuous compounding.

Problem Set 1.2:

1. Why is it important to introduce the constant of integration immediately when the integration is performed?

Find a general solution. Check your answer by substitution.

2. $y' + (x+1)y^3 = 0$
3. $y' = 3(y+1)$
4. $y' + \csc y = 0$
5. $y' = (1+x)(1+y^2)$
6. $yy' = \frac{1}{2}\sin^2 \omega x \quad (\omega \neq 0)$
7. $y' \sin 2x = y \cos 2x$
8. $y' = \cos x \tan y$
9. $y' = y \tanh x$
10. $y' + y^2 = 1$
11. $y' = e^{2x}\cos^2 y$
12. $y' = y^2 \sin x$
13. $y' = y/(x \ln x)$
14. $y' = x^2y^2 - 2y^2 + x^2 - 2$
15. $y' = \sqrt{1 - y^2}$

Solve the following initial value problems. Check your answer. (L and R are constant.)

16. $y' = x^3 e^{-y}, \quad y(2) = 0$
17. $yy' + x = 0, \quad y(0) = -2$
18. $y' = 2e^x y^3, \quad y(0) = 0.5$
19. $y' \cosh^2 x + \sin^2 y = 0, \quad y(0) = \frac{1}{4}\pi$
20. $y' = 4\sqrt{y+1}\cos 2x, \quad y(\frac{1}{4}\pi) = -1$
21. $y' = (1-x)/y, \quad y(1) = 1$
22. $dr/dt = -4tr, \quad r(0) = 8.2$
23. $v(dv/dt) = g = const, \quad v(t_0) = v_0$
24. $e^x y' = 2(x+3)y^3, \quad y(0) = \frac{1}{4}$
25. $dr \sin\theta = r\cos\theta\, d\theta, \quad r(\frac{1}{2}\pi) = -0.3$
26. $(x^2+1)^{1/2} y' = xy^3, \quad y(0) = 2$
27. $L(dI/dt) + RI = 0, \quad I(0) = I_0$

28. An initial value problem is usually solved by first determining the general solution of the equation and then the particular solution. Using (3), show that the particular solution of (1) satisfying the initial condition $y(x_0) = y_0$ can also be obtained directly from

$$\int_{y_0}^{y} g(y^*)\, dy^* = \int_{x_0}^{x} f(x^*)\, dx^*.$$

Using the formula in Prob. 28, solve:

29. Problem 17.
30. Problem 19.

Problem Set 1.4:

Find the general solution of the following equations.

1. $xy' = x + y$
2. $xy' = 2x + 2y$
3. $xyy' = \frac{1}{2}(y^2 + x^2)$
4. $x^2y' = y^2 + 5xy + 4x^2$
5. $x^2y' = y^2 + xy + x^2$
6. $(xy' - y)\cos(2y/x) = -3x^4$
7. $x^2y' = y^2 + xy$
8. $xy' = x\sec(y/x) + y$
9. $y' = \dfrac{y+x}{y-x}$
10. $y' = \dfrac{y-x}{y+x}$

Solve the following initial value problems.

11. $xy' = x + y, \quad y(1) = -7.4$
12. $xyy' = 2y^2 + 4x^2, \quad y(2) = 4$
13. $xy' = y + x^5e^x/4y^3, \quad y(1) = 0$
14. $xy' = y^2 + y, \quad y(4) = 2$
15. $yy' = x^3 + y^2/x, \quad y(2) = 6$
16. $xy' = y + x^2\sec(y/x), \quad y(1) = \pi/2$

Using the indicated transformation, find the general solution.

17. $y' = (y+x)^2 \quad (y + x = v)$
18. $y' = \tan(x+y) - 1 \quad (x + y = v)$
19. $2x^2yy' = \tan(x^2y^2) - 2xy^2 \quad (x^2y^2 = z)$
20. $y' = (x + e^y - 1)e^{-y} \quad (x + e^y = w)$
21. $y' = \dfrac{y - x + 1}{y - x + 5} \quad (y - x = v)$
22. $y' = \dfrac{1 - 2y - 4x}{1 + y + 2x} \quad (y + 2x = v)$

23. Consider $y' = f(ax + by + k)$, where f is continuous. If $b = 0$, the solution is immediate. (Why?) If $b \neq 0$, show that one obtains a separable equation by using $u(x) = ax + by + k$ as a new dependent variable.

Using Prob. 23, find the general solution of the following equations.

24. $y' = (x + y - 7)^2$
25. $y' = 2y + 8x$
26. $y' = (6y - y^2 - 8)^{1/2}$ (*Hint.* $u = y - 3$)
27. $y' = (5y + 2)^4$

28. Find the curve $y(x)$ that passes through $(1, 1/2)$ and is such that at each point (x, y) the intercept of the tangent on the y-axis is equal to $2xy^2$.
29. Show that a straight line through the origin intersects all solution curves of a given differential equation $y' = g(y/x)$ at the same angle.
30. The positions of four battle ships on the ocean are such that the ships form the vertices of a square of length l. At some instant each ship fires a missile that directs its motion steadily toward the missile on its right. Assuming that the four missiles fly horizontally and with the same speed, find the path of each.

Problem Set 1.5:

Given $u(x, y)$, find the exact differential equation $du = 0$. What kind of curves are the solution curves $u(x, y) = const$?

1. $x^2 + y^2 = c$
2. $u = y/x^2$
3. $u = (x - a)(b - y)$
4. $u = \cos(x^2 - y^2)$
5. $u = \exp(xy^2)$
6. $u = \sin xy$
7. $u = \ln(x^2y^2)$
8. $u = \tan(x^2 + 4y^2)$
9. $u = (y - x + 1)^2$
10. $u = \cosh(x^3 - y^2)$

Show that the following differential equations are exact and solve them.

11. $y\,dx + x\,dy = 0$
12. $x\,dx + 9y\,dy = 0$
13. $y^3\,dx + 3xy^2\,dy = 0$
14. $ye^x\,dx + [e^x + (y + 1)e^y]\,dy = 0$
15. $e^{-\theta}\,dr - re^{-\theta}\,d\theta = 0$
16. $\frac{1}{4}e^{4\theta}\,dr + re^{4\theta}\,d\theta = 0$
17. $\cosh x \cos y\,dx = \sinh x \sin y\,dy$
18. $e^x(\cos y\,dx - \sin y\,dy) = 0$
19. $(2x + e^y)\,dx + xe^y\,dy = 0$
20. $(\cot y + x^2)\,dx = x \operatorname{cosec}^2 y\,dy$

Are the following equations exact? Solve the initial value problems.

21. $x\,dy + y^2\,dx = 0, \quad y(1) = 0.2$
22. $4\,dx + x^{-1}\,dy = 0, \quad y(1) = -8$
23. $(y - 1)\,dx + (x - 3)\,dy = 0, \quad y(0) = 2/3$
24. $(x - 1)\,dx + (y + 1)\,dy = 0, \quad y(1) = 0$
25. $e^{y/x}(-y\,dx + x\,dy)/x^2 = 0, \quad y(-2) = -2$
26. $(2xy\,dx + dy)e^{x^2} = 0, \quad y(0) = 2$
27. $2xy\,dy = (x^2 + y^2)\,dx, \quad y(1) = 2$
28. $\cos \pi x \cos 2\pi y\,dx = 2 \sin \pi x \sin 2\pi y\,dy, \quad y(3/2) = 1/2$
29. $\sinh x\,dx + y^{-1} \cosh x\,dy = 0, \quad y(0) = \pi$
30. $2 \sin \omega y\,dx + \omega \cos \omega y\,dy = 0, \quad y(0) = \pi/2\omega$

31. Solve the equation in Example 3.
32. If an equation is separable, show that it is exact. Is the converse true?
33. Under what conditions is $(ax + by)\,dx + (kx + ly)\,dy = 0$ exact? (Here, a, b, k, l are constants.) Solve the exact equation.
34. Under what conditions is $[f(x) + g(y)]\,dx + [h(x) + p(y)]\,dy = 0$ exact?
35. Under what conditions is $f(x, y)\,dx + g(x)h(y)\,dy = 0$ exact?

To see that a differential equation can sometimes be solved by several methods, solve (a) by the present method, (b) by separation or inspection:

36. $xy' + y + 4 = 0$
37. $2x\,dx + x^{-2}(x\,dy - y\,dx) = 0$
38. $b^2x\,dx + a^2y\,dy = 0$
39. $3x^{-4}y\,dx = x^{-3}\,dy$

40. Can you figure out what the solution curves in Example 1 look like? *Hint.* Set $x = s + t$, $y = s - t$.

Problem Set 1.6:

1. Verify (4).
2. Verify exactness in Example 1 by the usual test.
3. Verify the solution in Example 2, as indicated.
4. Give the details of the derivation of (8).
5. Verify that Theorem 1 cannot be used to solve Example 4.
6. Verify that y, xy^3, and x^2y^5 are integrating factors of $y\,dx + 2x\,dy = 0$ and solve.

Verify that the given function F is an integrating factor and solve the initial value problem:

7. $2y\,dx + x\,dy = 0, \quad y(0.5) = 8, \quad F = x$
8. $3y\,dx + 2x\,dy = 0, \quad y(-1) = 1.4, \quad F = x^2y$
9. $(1 + xy)\,dx + x^2\,dy = 0, \quad y(1) = 0, \quad F = e^{xy}$
10. $dx + (x + y + 1)\,dy = 0, \quad y(2.5) = 0.5, \quad F = e^y$
11. $(2x^{-1}y - 3)\,dx + (3 - 2y^{-1}x)\,dy = 0, \quad y(1) = -1, \quad F = x^2y^2$
12. $y\,dx + [y + \tan(x + y)]\,dy = 0, \quad y(0) = \pi/2, \quad F = \cos(x + y)$
13. $y\,dx + [\coth(x - y) - y]\,dy = 0, \quad y(3) = 3, \quad F = \sinh(x - y)$
14. $(2xe^x - y^2)\,dx + 2y\,dy = 0, \quad y(0) = \sqrt{2}, \quad F = e^{-x}$

Find an integrating factor F and solve (using inspection or Theorems 1 and 2):

15. $2 \cos \pi y\,dx = \pi \sin \pi y\,dy$
16. $y \cos x\,dx + 3 \sin x\,dy = 0$
17. $(2y + xy)\,dx + 2x\,dy = 0$
18. $2y\,dx + 3x\,dy = 0$
19. $(1 + 2x^2 + 4xy)\,dx + 2\,dy = 0$
20. $2x\,dx = [3y^2 + (x^2 - y^3) \tan y]\,dy$
21. $(y + 1)\,dx - (x + 1)\,dy = 0$
22. $5\,dx - e^{y-x}\,dy = 0$
23. $ay\,dx + bx\,dy = 0$
24. $(3xe^y + 2y)\,dx + (x^2e^y + x)\,dy = 0$

In each case find conditions such that F is an integrating factor of (1). *Hint.* Assume $F(P\,dx + Q\,dy) = 0$ to be exact and apply (5) in Sec. 1.5.

25. $F = x^a$
26. $F = y^b$
27. $F = x^ay^b$
28. $F = e^y$

29. Using Prob. 27, derive the integrating factor in Prob. 11.
30. **(Checking)** Checking of solutions is always important. In connection with the present method it is particularly essential since one may have to exclude the function $y(x)$ given by $F(x, y) = 0$. To see this, consider $(xy)^{-1}\,dy - x^{-2}\,dx = 0$; show that an integrating factor is $F = y$ and leads to $d(y/x) = 0$, hence $y = cx$, where c is arbitrary, but $F = y = 0$ is not a solution of the original equation.

Problem Set 1.7:

1. Show that $e^{-\ln x} = 1/x$ (but not $-x$) and $e^{-\ln(\sec x)} = \cos x$.
2. Show that the choice of the value of the constant of integration in $\int p\,dx$ [see (4)] does not matter (so that we may choose it to be zero).
3. What is the limit of $y(t)$ as $t \to \infty$ in Example 2? Is it physically reasonable?

Find the general solutions of the following differential equations.

4. $y' + y = 5$
5. $y' - 4y = 0.8$
6. $y' + 2xy = 0$
7. $y' + 2y = 6e^x$
8. $y' - 2y = 2 - 4x$
9. $y' - 4y = 2x - 4x^2$
10. $y' = (y - 1) \cot x$
11. $xy' + 2y = 9x$
12. $y' \tan x = 2y$
13. $y' + 3y = e^{-3x}$
14. $y' + 2y = \cos x$
15. $xy' + 2y = 4e^{x^2}$
16. $(x^2 - 1)y' = xy$
17. $xy' - 2y = x^3e^x$
18. $x^2y' + 2xy = \sinh 3x$

Solve the following initial value problems.

19. $y' + 3y = 12, \quad y(0) = 6$

20. $y' = y \cot x, \quad y(\frac{1}{2}\pi) = 2$

21. $y' + y = (x + 1)^2, \quad y(0) = 3$

22. $y' + x^3 y = 4x^3, \quad y(0) = -1$

23. $y' + 2xy = 4x, \quad y(0) = 3$

24. $xy' = (1 + x)y, \quad y(1) = 3e$

25. $y' \coth 2x = 2y - 2, \quad y(0) = 0$

26. $y' = 2y/x + x^2 e^x, \quad y(2) = 0$

27. $y' + ky = e^{-kx}, \quad y(0) = 0.7$

28. $y' + 3x^2 y = xe^{-x^3}, \quad y(0) = -1$

General properties of linear differential equations. The *linear* differential equations (1) and (2) have certain important properties. In the next two chapters we shall see that the same is true for *linear* differential equations of higher order. This fact is quite important, since we can use it to obtain new solutions from given ones. Indeed, prove and illustrate with an example that the **homogeneous equation** (2) has the following properties.

29. $y = 0$ is a solution of (2), called the ***trivial solution***.

30. If y_1 is a solution of (2), then $y = cy_1$ (c any constant) is a solution of (2).

31. If y_1 and y_2 are solutions of (2), then their sum $y = y_1 + y_2$ is a solution of (2).

Prove and illustrate with an example that the **nonhomogeneous equation (1)** has the following properties.

32. If y_1 is a solution of (1) and y_2 is a solution of (2), then $y = y_1 + y_2$ is a solution of (1).

33. The difference $y = y_1 - y_2$ of two solutions y_1 and y_2 of (1) is a solution of (2).

34. If y_1 is a solution of (1), then $y = cy_1$ is a solution of $y' + py = cr$.

35. If y_1 is a solution of $y_1' + py_1 = r_1$ and y_2 is a solution of $y_2' + py_2 = r_2$ (with the same p), then $y = y_1 + y_2$ is a solution of $y' + py = r_1 + r_2$.

36. If $p(x)$ and $r(x)$ in (1) are constant, say, $p(x) = p_0$ and $r(x) = r_0$, then (1) can be solved by separating variables and the result will agree with that obtained from (4).

Reduction of nonlinear differential equations to linear form. Applying suitable transformations of variables, reduce to linear form and solve the following equations. *Hint.* Some are Bernoulli equations; some become linear if one takes y as the independent variable and x as the unknown function.

37. $y' + y = y^2$

38. $y' + y = x/y$

Reduction *(continued)*

39. $y' \cos y + x \sin y = 2x \quad (\sin y = z)$

40. $y' - 1 = e^{-y} \sin x$

41. $(e^y + x)y' = 1$

42. $y'(\sinh 3y - 2xy) = y^2$

43. $3y' + y = (1 - 2x)y^4$

44. $2xy' = 10x^3 y^5 + y$

45. $2xyy' + (x - 1)y^2 = x^2 e^x$

46. $y' \cos y + 2x \sin y = 2x$

Some applications (More in the next section)

47. How long will it take $y(t)$ in Example 2 to practically reach the limit, say, the value 399.9 lb? First guess.

48. Show that if in Example 2 we double the influx (but make no further changes), the model is $y' = 20 - [5/(200 + 5t)]y$, $y(0) = 40$. Solve this initial value problem.

49. If in Example 2 we replace the inflowing brine after 10 minutes by pure water (still flowing at 5 gal/min), how long will it take to get the tank practically salt-free, say, to decrease $y(t)$ to 0.01 lb? First guess.

50. **(Motion of a boat)** Two persons are riding in a motorboat, the combined weight of the persons and the boat being 4900 nt (about 1100 lb). Suppose that the motor exerts a constant force of 200 nt (about 45 lb) and the resistance R of the water is proportional to the speed v, say, $R = kv$ nt, where $k = 10$ nt · sec/meter. Set up the differential equation for $v(t)$, using Newton's second law

$$\text{Mass} \times \text{Acceleration} = \text{Force}.$$

Find $v(t)$ satisfying $v(0) = 0$. Find the maximum speed v_∞ at which the boat will travel (practically after a sufficiently long time). If the boat starts from rest, how long will it take to reach $0.9v_\infty$ and what distance does the boat travel during that time?

51. **(Newton's law of cooling)** Solve the differential equation in Example 2 of Sec. 1.3 by our present method, assuming the initial temperature of the ball to be $T(0) = T_0$.

52. **Hormone secretion** can be modeled by

$$y' = a - b \cos \frac{2\pi t}{24} - ky.$$

Here, t is time [in hours, with $t = 0$ suitably chosen, e.g., 8:00 A.M.], $y(t)$ is the amount of a certain hormone in the blood, a is the average secretion rate, $b \cos(\pi t/12)$ models the daily 24-hr secretion cycle, and ky models the removal rate of the hormone from the blood. Find the solution when $a = b = k = 1$ and $y(0) = 2$.

53. **(Logistic population model)** Show that (8) with $0 < y(0) < A/B$ grows monotone and with $y(0) > A/B$ decreases monotone.

54. **(United States)** For the United States, Verhulst predicted in 1845 the values $A = 0.03$ and $B = 1.6 \cdot 10^{-4}$, where x is measured in years and $y(x)$ in millions. Find the particular solution (8) satisfying $y(0) = 5.3$ (corresponding to the year 1800) and compare the values of this solution with some actual (rounded) values:

1800	1830	1860	1890	1920	1950	1980	1990
5.3	13	31	63	105	150	230	250

55. Show that the curves (8) have a point of inflection if $y(x) = A/2B$ [Use (7).]

Riccati and Clairaut equations

56. **A Riccati equation**[13] is of the form $y' + p(x)y = g(x)y^2 + h(x)$. Verify that the Riccati equation $y' = x^3(y - x)^2 + x^{-1}y$ has the solution $y = x$ and reduce it to a Bernoulli equation by the substitution $w = y - x$ and solve it.

57. Show that the general Riccati equation in Prob. 56 (which is a Bernoulli equation when $h = 0$) can be reduced to a Bernoulli equation if one knows a solution $y = v$, by setting $w = y - v$.

58. **A Clairaut equation**[14] is of the form $y = xy' + g(y')$. Solve the Clairaut equation $y = xy' + 1/y'$. *Hint.* Differentiate the equation with respect to x.

59. Show that the general Clairaut equation in Prob. 58, with arbitrary $g(s)$ has as solutions a family of straight lines $y = cx + g(c)$ and a singular solution determined by $g'(s) = -x$, where $s = y'$. (Those lines are tangents to the latter.) *Hint.* Differentiate the equation with respect to x, as in Prob. 58.

60. Show that the straight lines, whose segment between the positive x-axis and y-axis has constant length 1, are solutions of the Clairaut equation $y = xy' - y'/\sqrt{1 + y'^2}$, whose singular solution is the **astroid** $x^{2/3} + y^{2/3} = 1$. Make a sketch.

Problem Set 1.9:

Find the differential equation (6) of the following families.

1. $xy = c$	**2.** $e^{3x}y = c$	**3.** $y = c \sin 2x$
4. $y = cxe^x$	**5.** $y = e^{cx^2}$	**6.** $y = 1/(1 + ce^x)$
7. $y = cx^2 + x^2e^x$	**8.** $c^2x^2 + y^2 = c^2$	**9.** $y^2 = ce^{-2x} + x - \frac{1}{2}$

Using differential equations, find the orthogonal trajectories of the following curves. Graph some of the curves and the trajectories.

10. $x^2 - y^2 = c^2$	**11.** $x^2 + y^2 = c^2$	**12.** $y = c/x^2$
13. $y = \ln \lvert x \rvert + c$	**14.** $xy = c$	**15.** $y = \sqrt{x + c}$
16. $y = cx^3$	**17.** $x = ce^{-y^2}$	**18.** $x^2 + 2y^2 = c$
19. $x^2 + 4y^2 = c$	**20.** $(x - c)^2 + y^2 = c^2$	**21.** $y = ce^{8x}$

Applications

22. **(Temperature field)** If the **isotherms** (= curves of constant temperature) in a body are $T(x, y) = 2x^2 + y^2 = const$, what are their orthogonal trajectories (the curves along which heat will flow in regions free of heat sources or sinks and filled with homogeneous material)?

23. **(Electric field)** In the electric field between two concentric cylinders (Fig. 20) the **equipotential lines** (= curves of constant potential) are circles given by $U(x, y) = x^2 + y^2 = const$ [volts]. Use our present method to get their trajectories (the curves of electric force).

24. **(Electric field)** Experiments show that the electric lines of force of two opposite charges of the same strength at $(-1, 0)$ and $(1, 0)$ are the circles through $(-1, 0)$ and $(1, 0)$. Show that these circles can be represented by the equation $x^2 + (y - c)^2 = 1 + c^2$. Show that the equipotential lines (orthogonal trajectories) are the circles $(x + c^*)^2 + y^2 = c^{*2} - 1$ (dashed in Fig. 23).

25. **(Fluid flow)** If the **streamlines** of the flow (= paths of the particles of the fluid) in the channel in Fig. 24 are $\Psi(x, y) = xy = const$, what are their orthogonal trajectories (called **equipotential lines,** for reasons explained in Sec. 17.4)?

Other forms of the differential equations. Isogonal trajectories

26. Show that (7) can be written as $dx/dy = -f(x, y)$. Use it to get the orthogonal trajectories of the curves $y = x + ce^{-x}$.

27. Show that the orthogonal trajectories of a family $g(x, y) = c$ can be obtained from the following differential equation and use it to solve Prob. 25:

$$\frac{dy}{dx} = \frac{\partial g/\partial y}{\partial g/\partial x}.$$

28. **(Cauchy–Riemann equations).** Show that for a family $u(x, y) = c = const$ the orthogonal trajectories $v(x, y) = c^* = const$ can be obtained from the following so-called *Cauchy–Riemann equations* (which are basic in complex analysis in Chap. 12):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

29. Find the orthogonal trajectories of $e^x \cos y = c$ by the Cauchy–Riemann equations.

30. **Isogonal trajectories** of a given family of curves are curves that intersect the given curves at a constant angle θ. Show that at each point the slopes m_1 and m_2 of the tangents to the corresponding curves satisfy the relation

$$\frac{m_2 - m_1}{1 + m_1 m_2} = \tan\theta = const.$$

Using this, find the curves that intersect the circles $x^2 + y^2 = c$ at 45°.

Problem Set 2.1:

Important general properties of homogeneous and nonhomogeneous linear differential equations. Prove the following statements, which refer to any fixed open interval I, and illustrate them with examples. Here we assume that $r(x) \neq 0$ in (1).

1. $y \equiv 0$ is a solution of (2) (called the "*trivial solution*") but not of (1).
2. The sum of two solutions of (1) is **not** a solution of (1).
3. A multiple $y = cy_1$ of a solution of (1) is **not** a solution of (1), unless $c = 1$.
4. The sum $y = y_1 + y_2$ of a solution y_1 of (1) and y_2 of (2) is a solution of (1).
5. The difference $y = y_1 - y_2$ of two solutions of (1) is a solution of (2).

Second-order differential equations reducible to the first order

Problems 6–19 illustrate that certain second-order differential equations can be reduced to the first order.

6. If in a second-order equation the dependent variable y does not appear explicitly, the equation is of the form $F(x, y', y'') = 0$. Show that by setting $y' = z$ we obtain a first-order differential equation in z and from its solution the solution of the original equation by integration.

Reduce to the first order and solve:

7. $xy'' = 2y'$	**8.** $y'' = y'$	**9.** $y'' = 2y' \coth 2x$
10. $y'' + 9y' = 0$	**11.** $xy'' + y' = y'^2$	**12.** $xy'' + y' = 0$

13. Another type of equation reducible to first order is $F(y, y', y'') = 0$, in which the independent variable x does not appear explicitly. Using the chain rule, show that $y'' = (dz/dy)z$, where $z = y'$, so that we obtain a first-order equation with y as the *independent* variable.

Reduce to first order and solve:

14. $yy'' + y'^2 = 0$ **15.** $y'' + e^{2y}y'^3 = 0$ **16.** $y'' + y'^2 = 0$

17. $y'' + y'^3 \cos y = 0$ **18.** $yy'' - y'^2 = 0$ **19.** $y'' + (1 + y^{-1})y'^2 = 0$

20. A particle moves on a straight line so that the product of its velocity and acceleration is constant, say, 2 meters^2/sec^3. At time $t = 0$ its displacement from the origin is 5 meters and its velocity is zero. Find its position and velocity when $t = 9$ sec.

21. Find the curve in the xy-plane which passes through the point $(1, 1)$, intersects the line $y = x$ at a right angle, and satisfies $xy'' + 2y' = 0$.

22. (Hanging cable) It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called a *catenary* (from Latin *catena* = the chain). Find and graph $y(x)$, assuming $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.

Initial value problems

Verify that the given functions form a basis of solutions of the given equation and solve the given initial value problem.

23. $y'' - 9y = 0, \quad y(0) = 2, \quad y'(0) = 0; \quad e^{3x}, e^{-3x}$

24. $y'' + 4y = 0, \quad y(0) = -5, \quad y'(0) = 2; \quad \cos 2x, \sin 2x$

25. $y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = 3; \quad e^x, xe^x$

26. $y'' + \omega^2 y = 0 \quad (\omega \neq 0), \quad y(0) = -1.5, \quad y'(0) = 0; \quad \cos \omega x, \sin \omega x$

27. $y'' - 4y' + 3y = 0, \quad y(0) = -1, \quad y'(0) = -5; \quad e^x, e^{3x}$

28. $x^2y'' - 1.5xy' - 1.5y = 0, \quad y(1) = 1, \quad y'(1) = -4; \quad x^{-1/2}, x^3$

29. $x^2y'' - 3xy' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 5; \quad x^2, x^2 \ln x$

30. $y'' - 2y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 10; \quad e^x \cos 3x, e^x \sin 3x$

Are the following functions linearly dependent or independent on the given interval?

31. $x + 1, x - 1 \quad (0 < x < 1)$ **32.** $0, \cosh x$, any interval

33. $\sin 2x, \sin x \cos x$, any interval **34.** $\ln x, \ln x^2 \quad (x > 1)$

35. $|x|x, x^2 \quad (0 < x < 1)$ **36.** $|x|x, x^2 \quad (-1 < x < 1)$

37. $|\cos x|, \cos x \quad (0 < x < \pi)$ **38.** $x^2 - 3, -3x^2 + 9 \quad (x < 0)$

39. $\cosh x, \cosh 2x$, any interval **40.** $1, e^{-2x} \quad (x > 0)$

Problem Set 2.2:

Find a general solution of the following differential equations.

1. $y'' + 3y' + 2y = 0$ **2.** $y'' - 9y = 0$ **3.** $y'' + 10y' + 25y = 0$

4. $y'' + 4y' = 0$ **5.** $y'' - 6y' + 9y = 0$ **6.** $y'' - 2y' + 0.75y = 0$

Find the differential equation (1) for which the given functions form a basis of solutions.

7. e^{2x}, e^{-3x} **8.** $e^{-\pi x}, xe^{-\pi x}$ **9.** $e^{-2x}, e^{-x/2}$

10. e^{kx}, e^{-kx} **11.** $\cosh x, \sinh x$ **12.** $1, e^{-kx}$

13. Verify directly that in the case of a double root, $xe^{\lambda x}$ with $\lambda = -a/2$ is a solution of (1).

14. Verify that $y = e^{-3x}$ is a solution of $y'' + 5y' + 6y = 0$, but $y = xe^{-3x}$ is not. Explain.

15. Show that a and b in (1) can be expressed in terms of λ_1 and λ_2 by the formulas $a = -\lambda_1 - \lambda_2$, $b = \lambda_1\lambda_2$.

16. Solve $y'' + 3y' = 0$ (a) by the present method, (b) by reduction to the first order.

Solve the following initial value problems.

17. $y'' - 16y = 0, \quad y(0) = 1, \quad y'(0) = 20$

18. $y'' - 4y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = -3$

19. $y'' + 6y' + 9y = 0, \quad y(0) = -4, \quad y'(0) = 14$

20. $y'' + 3.7y' = 0, \quad y(-2) = 4, \quad y'(-2) = 0$

21. $y'' + 2.2y' + 0.4y = 0, \quad y(0) = 3.3, \quad y'(0) = -1.2$

22. $y'' - 25y = 0, \quad y(0) = 0, \quad y'(0) = 10$

23. $4y'' - 4y' - 3y = 0, \quad y(-2) = e, \quad y'(-2) = -\frac{1}{2}e$

24. $5y'' + 16y' + 12.8y = 0, \quad y(0) = 0, \quad y'(0) = -2.3$

25. Different bases lead to the same solution. To illustrate this, solve $y'' - 9y = 0$, $y(0) = 4$, $y'(0) = -6$, using (a) e^{3x}, e^{-3x}, (b) $\cosh 3x, \sinh 3x$.

Problem Set 2.3:

Verify that the following functions are solutions of the given differential equation and obtain from them a real-valued general solution of the form (6).

1. $y = c_1e^{3ix} + c_2e^{-3ix}, \quad y'' + 9y = 0$

2. $y = c_1e^{(-1+3i)x} + c_2e^{(-1-3i)x}, \quad y'' + 2y' + 10y = 0$

3. $y = c_1e^{-(\alpha+i)x} + c_2e^{-(\alpha-i)x}, \quad y'' + 2\alpha y' + (\alpha^2 + 1)y = 0$

4. $y = c_1e^{(-5+2i)x} + c_2e^{(-5-2i)x}, \quad y'' + 10y' + 29y = 0$

5. $y = c_1e^{-(\alpha-i\omega)x} + c_2e^{-(\alpha+i\omega)x}, \quad y'' + 2\alpha y' + (\alpha^2 + \omega^2)y = 0$

General Solution. State whether the given equation corresponds to Case I, Case II, or Case III and find a general solution involving real-valued functions.

6. $y'' + 25y = 0$ **7.** $y'' - 25y = 0$

8. $y'' - 8y' + 16y = 0$ **9.** $y'' + 6y' + 9y = 0$

10. $y'' + y' + 0.25y = 0$ **11.** $y'' + 2y' = 0$

12. $8y'' - 2y' - y = 0$ **13.** $10y'' + 6y' + 10.9y = 0$

14. $2y'' + 10y' + 25y = 0$ **15.** $y'' + 2y' + (\omega^2 + 1)y = 0$

Initial Value Problems. Solve the following initial value problems.

16. $y'' - 9y = 0, \quad y(0) = 5, \quad y'(0) = 9$

17. $y'' + 9y = 0, \quad y(\pi) = -2, \quad y'(\pi) = 3$

18. $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0$

19. $y'' - 4y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 10$
20. $y'' - 6y' + 18y = 0, \quad y(0) = 0, \quad y'(0) = 6$
21. $y'' + 20y' + 100y = 0, \quad y(0.1) = 3.2/e \approx 1.177, \quad y'(0.1) = -30/e \approx -11.04$
22. $10y'' + 2y' + 0.1y = 0, \quad y(10) = -40/e \approx -14.72, \quad y'(10) = 0$
23. $2y'' + y' - y = 0, \quad y(4) = e^2 - e^{-4} \approx 7.371, \quad y'(4) = \frac{1}{2}e^2 + e^{-4} \approx 3.713$
24. $y'' + 4y' + 4.25y = 0, \quad y(0) = 1, \quad y'(0) = -2$

Boundary Value Problems. Solve the following boundary value problems.
25. $y'' - 16y = 0, \quad y(0) = 5, \quad y(\frac{1}{4}) = 5e$
26. $y'' - 9y = 0, \quad y(-4) = y(4) = \cosh 12$
27. $y'' - 2y' = 0, \quad y(0) = -1, \quad y(\frac{1}{2}) = e - 2$
28. $y'' + 4y' + 5y = 0, \quad y(\frac{1}{2}\pi) = 14e^{-\pi} \approx 0.6050, \quad y(\frac{3}{2}\pi) = -14e^{-3\pi} \approx -0.0011$
29. $y'' - 2y' + 2y = 0, \quad y(0) = -3, \quad y(\frac{1}{2}\pi) = 0$
30. Show that the solution of a boundary value problem (1), (9) is unique if and only if no solution $y \neq 0$ of (1) satisfies $y(P_1) = y(P_2) = 0$.

Problem Set 2.4:

In each case apply the given operator to each of the given functions.
1. $D + 3; \quad x^2 + 6x - 2, \quad 9e^{-3x}, \quad \sin \pi x + 2\cos \pi x$
2. $D^2 - 2D; \quad xe^x, \quad \sinh 2x, \quad e^{2x} + 5$
3. $(D + 4)(D - 1); \quad e^{-4x}, \quad xe^{-4x}, \quad e^x$
4. $(D - 5)^2; \quad 5x + \cosh 5x, \quad e^{5x}, \quad xe^{5x}$

Find a general solution of the following equations.
5. $(D^2 + 2D + 2)y = 0$
6. $(4D^2 + 4D + 1)y = 0$
7. $(4D^2 - 12D + 9)y = 0$
8. $(D^2 + 6D + 12)y = 0$
9. $(\pi^2 D^2 - 4\pi D + 4)y = 0$
10. $(4D^2 + 4D + 17)y = 0$
11. $(10D^2 + 12D + 3.6)y = 0$
12. $(D^2 + 2kD + k^2 + 3)y = 0$

Solve the following initial value problems.
13. $(D^2 + 4D + 5)y = 0, \quad y(0) = 0, \quad y'(0) = -3$
14. $(D^2 + 5D + 6)y = 0, \quad y(0) = 2, \quad y'(0) = -3$
15. $(D^2 - 2D + \pi^2 + 1)y = 0, \quad y(0) = 1, \quad y'(0) = 1 - \pi$
16. $(D^2 - 0.1D - 3.8)y = 0, \quad y(0) = -3.9, \quad y'(0) = 7.8$
17. $(9D^2 + 6D + 1)y = 0, \quad y(-3) = 10e \approx 27.18, \quad y'(-3) = -\frac{19}{3}e \approx -17.22$
18. $(D^2 - 0.2D + 100.01)y = 0, \quad y(0) = 0, \quad y'(0) = 40$
19. $(D + 1)^2 y = 0, \quad y(0) = 1, \quad y'(0) = -2$
20. Prove that the operator L in (2) is linear.

Problem Set 2.6:

1. Verify directly by substitution that y_2 in (7*) is a solution of (1) if (3) has a double root, but $x^{m_1} \ln x$ and $x^{m_2} \ln x$ are **not** solutions of (1) if the roots m_1 and m_2 of (3) are different.

Find a general solution of the following differential equations.
2. $x^2y'' - 6y = 0$
3. $xy'' + 4y' = 0$
4. $x^2y'' - 2xy' + 2y = 0$
5. $(x^2D^2 + 9xD + 16)y = 0$
6. $x^2y'' + xy' - y = 0$
7. $(x^2D^2 + 3xD + 1)y = 0$
8. $(x^2D^2 - 1.5xD + 1)y = 0$
9. $x^2y'' + 6.2xy' + 6.76y = 0$
10. $x^2y'' + 3xy' + 5y = 0$
11. $(x^2D^2 + xD + 1)y = 0$
12. $(x^2D^2 - 3xD + 20)y = 0$
13. $(4x^2D^2 + 8xD - 15)y = 0$

Solve the following initial value problems.
14. $x^2y'' - 4xy' + 4y = 0, \quad y(1) = 4, \quad y'(1) = 13$
15. $(4x^2D^2 + 4xD - 1)y = 0, \quad y(4) = 2, \quad y'(4) = -0.25$
16. $(x^2D^2 - 5xD + 8)y = 0, \quad y(1) = 5, \quad y'(1) = 18$
17. $(x^2D^2 - xD + 2)y = 0, \quad y(1) = -1, \quad y'(1) = -1$
18. $10x^2y'' + 46xy' + 32.4y = 0, \quad y(1) = 0, \quad y'(1) = 2$
19. $(x^2D^2 + xD - 0.01)y = 0, \quad y(1) = 1, \quad y'(1) = 0.1$
20. **(Potential between two spheres)** Find the potential in Example 4 if $r_1 = 2$ cm, $r_2 = 20$ cm and the potentials on the spheres are $v_1 = 220$ volts and $v_2 = 130$ volts.
21. **(Equations with constant coefficients)** Setting $x = e^t$ $(x > 0)$, transform the Euler–Cauchy equation (1) into the equation $\ddot{y} + (a - 1)\dot{y} + by = 0$, whose coefficients are constant. Here, dots denote derivatives with respect to t.
22. Transform the equation in Prob. 21 back into (1).
23. Show that if we apply the transformation in Prob. 21, then (2) yields an expression of the form (2), Sec. 2.2, and (7) yields an expression of the form (7), Sec. 2.2, except for notation.

Reduce to the form (1) and solve:
24. $2(3z + 1)^2y'' + 21(3z + 1)y' + 18y = 0$
25. $(z - 2)^2y'' + 5(z - 2)y' + 3y = 0$

Problem Set 2.7:

Find the Wronskian of the following bases (that we have used before), thereby verifying Theorem 2 for any interval. (In Probs. 4–6, assume $x > 0$.)
1. $e^{\lambda_1 x}, e^{\lambda_2 x}, \quad \lambda_1 \neq \lambda_2$
2. $e^{\lambda x}, xe^{\lambda x}$
3. $e^{-ax/2}\cos \omega x, e^{-ax/2}\sin \omega x$
4. $x^{m_1}, x^{m_2}, \quad m_1 \neq m_2$
5. $x^{\mu}\cos(\nu \ln x), x^{\mu}\sin(\nu \ln x)$
6. $x^m, x^m \ln x$

Find a second-order homogeneous linear differential equation for which the given functions are solutions. Find the Wronskian and use it to verify Theorem 2.

7. e^x, xe^x
8. $x, x \ln x$
9. $e^x \cos x, e^x \sin x$
10. $\cosh kx, \sinh kx$
11. $\cos \pi x, \sin \pi x$
12. $\sqrt{x}, 1/\sqrt{x}$
13. $1, x^3$
14. $1, e^{-2x}$
15. $x^{1/2}, x^{3/2}$

16. Suppose that (1) has continuous coefficients on I. Show that two solutions of (1) on I that are zero at the same point in I cannot form a basis of solutions of (1) on I.

17. Suppose that (1) has continuous coefficients on I. Show that two solutions of (1) on I that have maxima or minima at the same point in I cannot form a basis of solutions of (1) on I.

18. Suppose that y_1, y_2 constitute a basis of solutions of a differential equation satisfying the assumptions of Theorem 2. Show that $z_1 = a_{11}y_1 + a_{12}y_2$, $z_2 = a_{21}y_1 + a_{22}y_2$ is a basis of that equation on the interval I if and only if the determinant of the coefficients a_{jk} is not zero.

19. Illustrate Prob. 18 with $y_1 = e^x$, $y_2 = e^{-x}$, $z_1 = \cosh x$, $z_2 = \sinh x$.

20. **(Euler–Cauchy equation)** Show that $x^2y'' - 4xy' + 6y = 0$ (Sec. 2.6) has $y_1 = x^2$, $y_2 = x^3$ as a basis of solutions for all x. Show that $W(x^2, x^3) = 0$ at $x = 0$. Does this contradict Theorem 2?

Reduction of order. Show that the given function y_1 is a solution of the given equation. Using the method of reduction of order, find y_2 such that y_1, y_2 form a basis. ***Caution!*** First write the equation in standard form.

21. $(x+1)^2y'' - 2(x+1)y' + 2y = 0, \quad y_1 = x + 1$
22. $(x-1)y'' - xy' + y = 0, \quad y_1 = e^x$
23. $(1-x)^2y'' - 4(1-x)y' + 2y = 0, \quad y_1 = (1-x)^{-1}, \quad x \neq 1$
24. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0, \quad y_1 = x^{-1/2} \cos x, \quad x > 0$
25. $xy'' + 2y' + xy = 0, \quad y_1 = x^{-1} \sin x, \quad x \neq 0$

Problem Set 2.9:

Find a general solution of the following differential equations.

1. $y'' + y = 3x^2$
2. $y'' - 4y = e^{2x}$
3. $y'' + 6y' + 9y = 18 \cos 3x$
4. $y'' + 4y' + 4y = 9 \cosh x$
5. $y'' - y' - 2y = e^x + x$
6. $y'' - 2y' + 2y = 2e^x \cos x$
7. $y'' + 2y' + 5y = 5x^2 + 4x + 2$
8. $3y'' + 10y' + 3y = x^2 + \sin x$
9. $(D^2 - 4D + 3)y = 2 \sin x - 4 \cos x$
10. $(D^2 + 5D + 6)y = 9x^4 - x$
11. $(D^2 - 2D + 1)y = 2e^x$
12. $(D^2 - 4D + 3)y = 8e^{-3x} + e^{3x}$
13. $(D^2 + 9)y = \cos 3x$
14. $(D^2 + 5D)y = 1 + x + e^x$
15. $(D^2 - 4)y = 2 \sinh 2x + x$
16. $(D^2 + D - 6)y = 52 \cos 2x$

Problem Set 2.10:

Find a general solution of the following equations.

1. $y'' - 2y' + y = x^{3/2}e^x$
2. $y'' + 4y = 2 \sec 2x$
3. $y'' - 2y' + y = 12e^x/x^3$
4. $y'' + y = \csc x$
5. $y'' + 4y' + 4y = e^{-2x}/x^2$
6. $y'' - 4y' + 5y = 2e^{2x}/\sin x$
7. $y'' - 2y' + y = 35x^{3/2}e^x + x^2$
8. $y'' - 4y' + 4y = (3x^2 + 2)e^x$
9. $y'' - 2y' + y = e^x \sin x$
10. $y'' + 6y' + 9y = 8e^{-3x}/(x^2 + 1)$
11. $(D^2 - 4D + 4)y = 6x^{-4}e^{2x}$
12. $(D^2 + 9)y = \sec 3x$
13. $(D^2 - 2D + 1)y = e^x/x^3$
14. $(D^2 + 2D + 2)y = e^{-x}/\cos^3 x$

Nonhomogeneous Euler–Cauchy Equations. Find a general solution of the following equations. ***Caution!*** First divide the equation by the coefficient of y'' to get the standard form (1).

15. $(x^2D^2 - 4xD + 6)y = 42/x^4$
16. $(x^2D^2 - 2)y = 9x^2$
17. $(x^2D^2 - 2xD + 2)y = 5x^3 \cos x$
18. $(xD^2 - D)y = (3 + x)x^2e^x$
19. $(xD^2 - D)y = x^2e^x$
20. $(x^2D^2 - 4xD + 6)y = -7x^4 \sin x$
21. $x^2y'' - 2xy' + 2y = 24/x^2$
22. $4x^2y'' + 4xy' - y = 12/x$
23. $x^2y'' - 4xy' + 6y = 1/x^4$
24. $x^2y'' - 2xy' + 2y = x^4$

25. **(Comparison of methods)** Whenever the method of undetermined coefficients (Sec. 2.9) is applicable, it should be used because it is simpler than the present method. To illustrate this fact, solve by both methods

$$y'' + 4y' + 3y = 65 \cos 2x.$$

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Problem Set 3.5:

Find a general solution.

1. $y''' - 3y'' + 3y' - y = x^{1/2}e^x$
2. $y''' - 6y'' + 12y' - 8y = \sqrt{2x}\, e^{2x}$
3. $y''' - 6y'' + 11y' - 6y = e^{2x} \sin x$
4. $y''' - y' = \cosh x$
5. $y''' + y' = \sec x$
6. $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \sinh x$
7. $xy''' + 3y'' = e^x$
8. $x^3y''' + x^2y'' - 2xy' + 2y = x^3 \ln x$
9. $x^3y''' + x^2y'' - 2xy' + 2y = x^{-2}$
10. $4x^3y''' + 3xy' - 3y = 4x^{11/2}$

Respostas:

PROBLEM SET 1.1, page 8

1. First order 3. First order 5. First order
7. $-\frac{1}{2}e^{-2x} + c$ 9. $-2 \sin \frac{1}{2}x + c$ 11. $1.5e^{-x} + 1$
13. $3x^2$ 15. $-2 \sin x$ 17. $y' = 2y/x$
19. $y' = y^2 + 1$
23. $y'' = k$, $y = \frac{1}{2}kt^2 + 5t$, $y(40) = 800k + 200 = 1800$, $k = 2$, $y'(40) = 85$ m/sec $= 190$ mi/hr
25. (a) $3e^x$, (b) $y(2) = ce^2 = 100$, $y(0) = c = 13.53$
27. $y = 5.3 \exp(0.030t) = 32$ (1860), 78 (1890), the other values being much too large. A better model is the "logistic law" in Sec. 1.7.
29. About 1600 years

PROBLEM SET 1.2, page 13

1. Suppose you forgot c in Example 2, wrote arc tan $y = x$, transformed this to $y = \tan x$ and afterward added a constant c. Then $y = \tan x + c$, which, for $c \neq 0$, is not a solution of $y' = 1 + y^2$.
3. $y = ce^{3x} - 1$ 5. $y = \tan(x + \frac{1}{2}x^2 + c)$
7. $y = c(\sin 2x)^{1/2}$ 9. $y = c \cosh x$
11. $\tan y = \frac{1}{2}e^{2x} + c$ 13. $y = c \ln |x|$
15. $y = \sin(x + c)$ 17. $y = -\sqrt{4 - x^2}$
19. $\cot y = \tanh x + 1$ 21. $(x - 1)^2 + y^2 = 1$
23. $v^2 = v_0^2 + 2g(t - t_0)$ 25. $r = -0.3 \sin \theta$
27. $I = I_0 e^{-(R/L)t}$

PROBLEM SET 1.4, page 22

1. $y = x(\ln |x| + c)$ 3. $y^2 = x^2 - cx$
5. $y = x \tan(\ln |x| + c)$ 7. $y = -x/(\ln |x| + c)$
9. $y^2 - 2xy - x^2 = c$ 11. $y = x(\ln |x| - 7.4)$
13. $y = x(e^x - e)^{1/4}$ 15. $y = x(x^2 + 5)^{1/2}$
17. $y = -x + \tan(x + c)$ 19. $\sin(x^2y^2) = ce^x$
21. $(y - x)^2 + 10y - 2x = c$
23. $u' = a + by' = a + bf(u)$, $du/(a + bf(u)) = dx$
25. $y = ce^{2x} - 4x - 2$ 27. $y = [(c - 15x)^{-1/3} - 2]/5$
29. $y = ax$, $y/x = a = const$, $y' = g(a) = const$

PROBLEM SET 1.5, page 26

1. $2x\,dx + 2y\,dy = 0$ 3. $(b - y)\,dx - (x - a)\,dy = 0$
5. $e^{xy^2}(y^2\,dx + 2xy\,dy) = 0$ 7. $2(y\,dx + x\,dy)/xy = 0$
9. $2(y - x + 1)(-dx + dy) = 0$ 11. $xy = c$
13. $xy^3 = c$ 15. $re^{-\theta} = c$
17. $\sinh x \cos y = c$ 19. $x(x + e^y) = c$
21. No, $y = 1/(5 + \ln |x|)$ 23. Yes, $(x - 3)(y - 1) = 1$
25. Yes, $e^{y/x} = e$ or $y = x$ 27. No, $y = \sqrt{x^2 + 3x}$. Use $u = y/x$.
29. No, $y = \pi/\cosh x$ 31. $y = cx$
33. $b = k$, $ax^2 + 2kxy + ly^2 = c$ 35. $\partial f/\partial y = g'(x)h(y)$
37. $x^2 + y/x = c$ 39. $y = cx^3$

PROBLEM SET 1.6, page 29

Comment. Since integrating factors are not unique, your F's may differ from those given here.

7. $x^2y = 2$ 9. $xe^{xy} = 1$ 11. $x^2y^3 - x^3y^2 = -2$
13. $y \cosh(x - y) = 3$ 15. $F = e^{2x}$, $e^{2x} \cos \pi y = c$
17. $F = 1/xy$, $x^2y^2e^x = c$ 19. $F = e^{x^2}$, $(x + 2y)e^{x^2} = c$
21. $F = 1/(x + 1)(y + 1)$, $y + 1 = c(x + 1)$
23. $F = x^{a-1}y^{b-1}$, $x^ay^b = c$ 25. $\partial P/\partial y = \partial Q/\partial x + (a/x)Q$
27. $(b/y)P + \partial P/\partial y = (a/x)Q + \partial Q/\partial x$
29. $\frac{b}{y}\left(\frac{2y}{x} - 3\right) + \frac{2}{x} = \frac{a}{x}\left(3 - \frac{2x}{y}\right) - \frac{2}{y}$. Now equate the terms in $1/x$ and in $1/y$ to get $a = b = 2$.

PROBLEM SET 1.7, page 35

5. $y = ce^{4x} - 0.2$ 7. $y = ce^{-2x} + 2e^x$ 9. $y = ce^{4x} + x^2$
11. $y = cx^{-2} + 3x$ 13. $y = (c + x)e^{-3x}$ 15. $y = (c + 2e^{x^2})x^{-2}$
17. $y = cx^2 + x^2e^x$ 19. $y = 2e^{-3x} + 4$ 21. $y = 2e^{-x} + x^2 + 1$
23. $y = e^{-x^2} + 2$ 25. $y = 1 - \cosh 2x$ 27. $y = (0.7 + x)e^{-kx}$
37. $y = 1/(1 + ce^x)$ 39. $\sin y = 2 + ce^{-x^2/2}$ 41. $x = (c + y)e^y$
43. $y^{-3} = ce^x - 2x - 1$ 45. $y^2 = cxe^{-x} + \frac{1}{2}xe^x$ 47. 327.5 min
49. $y' + 0.025y = 0$, $y = \tilde{c}e^{-0.025t}$ if $t \geqq 10$, $y(10) = \tilde{c}e^{-0.25} = 119.6$ from Example 2, $\tilde{c} = 153.6$, $y(t) = 0.01$ if $t = 385.6$ min.
51. $T(t) = T_1 + (T_0 - T_1)e^{-kt}$, where T_1 is the temperature of the water
53. Eq. (7) gives $y' = Ay[1 - (B/A)y] > 0$ if $0 < y < A/B$ and $y' < 0$ if $y > A/B$.
55. Eq. (7) gives $y'' = Ay' - 2Byy' = 0$ if $A - 2By = 0$.
57. Substitute $y = w + v$, $y' = w' + v'$. Since v is a solution, there remains $w' = -pw + g(w^2 + 2wv)$, a Bernoulli equation.
59. $y' = y' + xy'' + (dg/ds)y''$, $y''(x + dg/ds) = 0$, etc.

PROBLEM SET 1.9, page 46

1. $xy' + y = 0$ 3. $y' = 2y \cot 2x$ 5. $xy' = 2y \ln |y|$
7. $xy' - 2y = x^3e^x$ 9. $y' + y = x/y$ 11. $y = c^*x$
13. $y = c^* - \frac{1}{2}x^2$ 15. $y = c^*e^{-2x}$ 17. $y = c^*e^{x^2}$
19. $y = c^*x^4$ 21. $y = \frac{1}{2}\sqrt{c - x}$ 23. $y = c^*x$
25. $x^2 - y^2 = c^*$ 29. $e^x \sin y = c^*$

PROBLEM SET 2.1, page 67

3. Since $y'' + py' + qy = cr$
5. Since $y'' + py' + qy = r - r$
7. $y = c_1x^3 + c_2$
9. $y = c_1 \cosh 2x + c_2$
11. $y = c_1^{-1} \ln |c_1x + 1| + c_2$
15. $x = \frac{1}{4}e^{2y} + c_1y + c_2$
17. $x = -\cos y + c_1y + c_2$
19. $(y - 1)e^y = c_1x + c_2$
21. $xy = 1$
23. $y = e^{3x} + e^{-3x} = 2 \cosh 3x$
25. $y = (4 - x)e^x$
27. $y = e^x - 2e^{3x}$
29. $y = (2 + \ln |x|)x^2$
31. Linearly independent
33. Linearly dependent
35. Linearly dependent
37. Linearly independent
39. Linearly independent

PROBLEM SET 2.2, page 72

1. $y = c_1e^{-x} + c_2e^{-2x}$
3. $y = (c_1 + c_2x)e^{-5x}$
5. $y = (c_1 + c_2x)e^{3x}$
7. $y'' + y' - 6y = 0$
9. $y'' + 2.5y' + y = 0$
11. $y'' - y = 0$
15. $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 + a\lambda + b$. Now compare.
17. $y = 3e^{4x} - 2e^{-4x}$
19. $y = (2x - 4)e^{-3x}$
21. $y = 3e^{-0.2x} + 0.3e^{-2x}$
23. $y = e^{-0.5x}$
25. $y = e^{3x} + 3e^{-3x} = 4 \cosh 3x - 2 \sinh 3x$

PROBLEM SET 2.3, page 76

1. $y = A \cos 3x + B \sin 3x$
3. $y = e^{-\alpha x}(A \cos x + B \sin x)$
5. $y = e^{-\alpha x}(A \cos \omega x + B \sin \omega x)$
7. I, $y = c_1e^{5x} + c_2e^{-5x}$
9. II, $y = (c_1 + c_2x)e^{-3x}$
11. I, $y = c_1 + c_2e^{-2x}$
13. III, $e^{-0.3x}(A \cos x + B \sin x)$
15. III, $y = e^{-x}(A \cos \omega x + B \sin \omega x)$
17. $y = 2 \cos 3x - \sin 3x$
19. $y = (3 + 4x)e^{2x}$
21. $y = (3 + 2x)e^{-10x}$
23. $y = e^{0.5x} - e^{-x}$
25. $y = 5e^{4x}$
27. $y = e^{2x} - 2$
29. $y = -3e^x \cos x$

PROBLEM SET 2.4, page 79

1. $3x^2 + 20x, 0, (6 + \pi) \cos \pi x + (3 - 2\pi) \sin \pi x$
3. $0, -5e^{-4x}, 0$
5. $y = e^{-x}(A \cos x + B \sin x)$
7. $y = (c_1 + c_2x)e^{3x/2}$
9. $y = (c_1 + c_2x)e^{2x/\pi}$
11. $y = (c_1 + c_2x)e^{-0.6x}$
13. $y = -3e^{-2x} \sin x$
15. $y = e^x(\cos \pi x - \sin \pi x)$
17. $y = (1 - 3x)e^{-x/3}$
19. $y = (1 - x)e^{-x}$

PROBLEM SET 2.6, page 93

3. $y = c_1 + c_2x^{-3}$
5. $y = (c_1 + c_2 \ln x)x^{-4}$
7. $y = (c_1 + c_2 \ln x)/x$
9. $y = (c_1 + c_2 \ln x)x^{-2.6}$
11. $y = A \cos (\ln x) + B \sin (\ln x)$
13. $y = c_1x^{1.5} + c_2x^{-2.5}$
15. $y = 4/\sqrt{x}$
17. $y = -x \cos (\ln x)$
19. $y = x^{0.1}$
21. $x = e^t$, $t = \ln x$; by the chain rule, $y' = \dot{y}t' = \dot{y}/x$, $y'' = \ddot{y}/x^2 - \dot{y}/x^2$, hence $\ddot{y} - \dot{y} + a\dot{y} + by = 0$.
25. $z - 2 = x$, $y = c_1(z - 2)^{-3} + c_2(z - 2)^{-1}$

PROBLEM SET 2.7, page 98

1. $W = (\lambda_2 - \lambda_1) \exp (\lambda_1 x + \lambda_2 x)$
3. $W = e^{-\alpha x}\omega$
5. $W = \nu x^{2\mu - 1}$
7. $y'' - 2y' + y = 0$, $W = e^{2x}$
9. $y'' - 2y' + 2y = 0$, $W = e^{2x}$
11. $y'' + \pi^2 y = 0$, $W = \pi$
13. $xy'' - 2y' = 0$, $W = 3x^2$
15. $x^2y'' - xy' + 0.75y = 0$, $W = x$
17. $W = 0$, since $y_1' = 0$, $y_2' = 0$ at such an x, by calculus.
19. $z_1 = (y_1 + y_2)/2$, $z_2 = (y_1 - y_2)/2$, $\det [a_{jk}] = -1/2$
21. $y_2 = x^2 + x$
23. $y_2 = (1 - x)^{-2}$
25. $y_2 = x^{-1} \cos x$

PROBLEM SET 2.9, page 105

1. $A \cos x + B \sin x + 3x^2 - 6$
3. $(c_1 + c_2x)e^{-3x} + \sin 3x$
5. $c_1e^{-x} + c_2e^{2x} - \frac{1}{2}e^x - \frac{1}{2}x + \frac{1}{4}$
7. $e^{-x}(A \cos 2x + B \sin 2x) + x^2$
9. $c_1e^x + c_2e^{3x} + \sin x$
11. $(c_1 + c_2x)e^x + x^2e^x$
13. $A \cos 3x + B \sin 3x + \frac{1}{6}x \sin 3x$
15. $(c_1 + \frac{1}{4}x)e^{2x} + (c_2 + \frac{1}{4}x)e^{-2x} - \frac{1}{4}x$
17. $e^{-x} - e^{2x} + xe^{2x}$
19. $4e^x + 2e^{-2x} + x^2 - 6$
21. $\cos x - 3 \sin x$
23. $3 \cos 2x - e^{-2x}$
25. $c_1e^{-2x} + \frac{1}{4} \cos 2x + \frac{1}{4} \sin 2x$

PROBLEM SET 2.10, page 109

1. $(c_1 + c_2x + \frac{4}{35}x^{7/2})e^x$
3. $(c_1 + c_2x)e^x + 6e^x/x$
5. $(c_1 + c_2x - \ln |x|)e^{-2x}$
7. $(c_1 + c_2x + 4x^{7/2})e^x + x^2 + 4x + 6$
9. $(c_1 + c_2x - \sin x)e^x$
11. $(c_1 + c_2x + x^{-2})e^{2x}$
13. $(c_1 + c_2x + (2x)^{-1})e^x$
15. $c_1x^2 + c_2x^3 + x^{-4}$
17. $c_1x + c_2x^2 - 5x \cos x$
19. $c_1 + c_2x^2 + (x - 1)e^x$
21. $c_1x + c_2x^2 + 2x^{-2}$
23. $c_1x^2 + c_2x^3 + \frac{1}{42}x^{-4}$
25. $c_1e^{-x} + c_2e^{-3x} + 8 \sin 2x - \cos 2x$

PROBLEM SET 3.5, page 149

1. $[c_1 + c_2x + c_3x^2 + (8/105)x^{7/2}]e^x$
3. $c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}e^{2x} \cos x$
5. $c_1 + c_2 \cos x + c_3 \sin x + \ln |\sec x + \tan x| - x \cos x + (\sin x) \ln |\cos x|$
7. $c_1x^{-1} + c_2 + c_3x + x^{-1}e^x$
9. $c_1x^{-1} + c_2x + c_3x^2 - 1/12x^2$