Problem Set 5.2:

Apply the power series method to the following differential equations.

1. $xy' = 3y + 3$
2. $(x - 3)y' - xy = 0$
3. $y' = 2xy$
4. $(1 - x^4)y' = 4x^3y$
5. $(x + 1)y' - (2x + 3)y = 0$
6. $y'' - y = x$
7. $y'' - 3y' + 2y = 0$
8. $y'' - 4xy' + (4x^2 - 2)y = 0$
9. $(1 - x^2)y'' - 2xy' + 2y = 0$
10. $y'' - xy' + y = 0$

11. Show that $y' = (y/x) + 1$ cannot be solved for y as a power series in x. Solve this equation for y as a power series in powers of $x - 1$. (*Hint*. Introduce $t = x - 1$ as a new independent variable and solve the resulting equation for y as a power series in t.) Compare the result with that obtained by the appropriate elementary method.

Solve for y as a power series in powers of $x - 1$:

12. $y' = ky$
13. $y'' + y = 0$
14. $y'' - y = 0$

Radius of convergence. Find the radius of convergence of the following series.

15. $\sum_{m=0}^{\infty} \frac{x^m}{3^m}$
16. $\sum_{m=0}^{\infty} (-1)^m x^{2m}$
17. $\sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}$
18. $\sum_{m=2}^{\infty} \frac{m(m-1)}{3^m} x^m$
19. $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$
20. $\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$
21. $\sum_{m=0}^{\infty} \left(\frac{7}{5}\right)^m x^{2m}$
22. $\sum_{m=0}^{\infty} \frac{(-1)^m}{5^m} (x - 5)^m$
23. $\sum_{m=0}^{\infty} m^{2m} x^m$
24. $\sum_{m=1}^{\infty} \frac{1}{3^m m^2} (x + 1)^m$
25. $\sum_{m=0}^{\infty} \frac{1}{2^m} (x - 1)^{2m}$
26. $\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m$

27. **(Shift of summation index)** Show that

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{j=1}^{\infty} (j+1) j a_{j+1} x^{j-1} = \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s.$$

Shift of summation index. Shift the index so that the power under the summation sign is x^m. Check your result by writing the first few terms explicitly. Also determine the radius of convergence.

28. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n} x^{n+2}$
29. $\sum_{s=2}^{\infty} \frac{s(s+1)}{s^2+1} x^{s-1}$
30. $\sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{6^k} x^{k-3}$

Problem Set 5.3:

1. Using (11′), verify by substitution that $P_0, \cdots, P_5$ satisfy Legendre's equation.
2. Find and graph $P_6(x)$.
3. Derive (11′) from (11).
4. Show that we can get (3) from (1*) more quickly if we write $m - 2 = s$ in the first sum in (1*) and $m = s$ in the other sums, obtaining

$$\sum_{s=0}^{\infty} \{(s+2)(s+1)a_{s+2} - [s(s-1) + 2s - k]a_s\}x^s = 0.$$

5. Show that for any n for which the series (6) or (7) do not reduce to a polynomial, these series have radius of convergence 1.
6. Solve (1) with $n = 0$ as indicated in Example 1.
7. Using (11), show that $P_n(-x) = (-1)^n P_n(x)$ and $P_n'(-x) = (-1)^{n+1} P_n'(x)$.
8. **(Rodrigues's formula[5])** Applying the binomial theorem to $(x^2 - 1)^n$, differentiating n times term by term, and comparing with (11), show that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n] \qquad (12) \quad \textit{(Rodrigues's formula)}.$$

9. Using (12) and integrating n times by parts, show that

$$\int_{-1}^{1} P_n^{\,2}(x)\,dx = \frac{2}{2n+1} \qquad (13) \quad (n = 0, 1, \cdots).$$

10. **(Generating function)** Show that

$$\frac{1}{\sqrt{1 - 2xu + u^2}} = \sum_{n=0}^{\infty} P_n(x)u^n. \qquad (14)$$

The function on the left is called a *generating function* of the Legendre polynomials. *Hint*. Start from the binomial expansion of $1/\sqrt{1 - v}$, then set $v = 2xu - u^2$, multiply the powers of $2xu - u^2$ out, collect all the terms involving u^n, and verify that the sum of these terms is $P_n(x)u^n$.

11. Let A_1 and A_2 be two points in space (Fig. 83, $r_2 > 0$). Using (14), show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1 r_2 \cos\theta}} = \frac{1}{r_2} \sum_{m=0}^{\infty} P_m(\cos\theta) \left(\frac{r_1}{r_2}\right)^m.$$

This formula has applications in potential theory.

Using (14), show that

12. $P_n(1) = 1$ **13.** $P_n(-1) = (-1)^n$

14. $P_{2n+1}(0) = 0$ **15.** $P_{2n}(0) = (-1)^n \cdot 1 \cdot 3 \cdots (2n-1)/2 \cdot 4 \cdots (2n)$.

16. **(Bonnet's recursion[6])** Differentiating (14) with respect to u, using (14) in the resulting formula, and comparing coefficients of u^n, obtain the *Bonnet recursion*

$$(15) \qquad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n = 1, 2, \cdots.$$

17. **(Computation)** Formula (15) is useful for computations, the loss of significant figures being small (except at zeros). Using (15), compute $P_2(2.6)$ and $P_3(2.6)$.

18. Using (15) and (11′), find P_6.

19. **(Associated Legendre functions)** Consider

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0.$$

Substituting $y(x) = (1-x^2)^{m/2}u(x)$, show that u satisfies

$$(16) \qquad (1-x^2)u'' - 2(m+1)xu' + [n(n+1) - m(m+1)]u = 0.$$

Starting from (1) and differentiating it m times, show that a solution of (16) is

$$u = \frac{d^mP_n}{dx^m}.$$

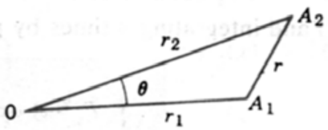

Fig. 83. Problem 11

The corresponding $y(x)$ is denoted by $P_n^m(x)$. It is called an *associated Legendre function* and plays a role in quantum physics. Thus

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^mP_n}{dx^m}.$$

20. Find $P_1^1(x)$, $P_2^1(x)$, $P_2^2(x)$, $P_4^2(x)$.

Problem Set 5.4:

Find a basis of solutions of the following differential equations. Try to identify the series obtained by the Frobenius method as expansions of known functions.

1. $xy'' + 2y' + xy = 0$ **2.** $xy'' + 2y' + 4xy = 0$

3. $x^2y'' + 6xy' + (6 - x^2)y = 0$ **4.** $16x^2y'' + 3y = 0$

5. $x^2y'' + 6xy' + (x^2+6)y = 0$ **6.** $(x+2)^2y'' + (x+2)y' - y = 0$

7. $2x^2y'' + xy' - 3y = 0$ **8.** $x^2y'' + x^3y' + (x^2-2)y = 0$

9. $x^2y'' + 4xy' + (x^2+2)y = 0$ **10.** $(x+1)^2y'' + (x+1)y' - y = 0$

11. $xy'' + (1-2x)y' + (x-1)y = 0$ **12.** $xy'' + (2-2x)y' + (x-2)y = 0$

13. $xy'' + 3y' + 4x^3y = 0$ **14.** $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

15. $(x-1)^2y'' + (x-1)y' - 4y = 0$ **16.** $xy'' + y' - xy = 0$

17. $(1+x)x^2y'' - (1+2x)xy' + (1+2x)y = 0$

18. $(1+\frac{1}{2}x)x^2y'' - (1+x)xy' + (1+x)y = 0$

19. $2x(x-1)y'' - (4x^2-3x+1)y' + (2x^2-x+2)y = 0$

20. $(x^2-1)x^2y'' - (x^2+1)xy' + (x^2+1)y = 0$

Hypergeometric equation, hypergeometric series, hypergeometric functions

21. **Gauss's hypergeometric differential equation[11]** is

$$(17) \qquad x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

where a, b, c are constants. Show that the corresponding indicial equation has the roots $r_1 = 0$ and $r_2 = 1 - c$. Show that for $r_1 = 0$ the Frobenius method yields

$$(18) \qquad y_1(x) = 1 + \frac{ab}{1!\,c}x + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \cdots$$

where $c \neq 0, -1, -2, \cdots$. This series is called the **hypergeometric series.** Its sum $y_1(x)$ is denoted by $F(a, b, c; x)$ and is called the **hypergeometric function.**

Using (18), prove:

22. The series (18) converges for $|x| < 1$.

23. $F(1, b, b; x) = 1 + x + x^2 + \cdots$, the geometric series.

24. If a or b is a negative integer, (18) reduces to a polynomial.

25. $\dfrac{dF(a,b,c;x)}{dx} = \dfrac{ab}{c}F(a+1, b+1, c+1; x)$,

$\dfrac{d^2F(a,b,c;x)}{dx^2} = \dfrac{a(a+1)b(b+1)}{c(c+1)}F(a+2, b+2, c+2; x)$, etc.

Many elementary functions are special cases of $F(a, b, c; x)$. Prove

26. $\dfrac{1}{1-x} = F(1, 1, 1; x) = F(1, b, b; x) = F(a, 1, a; x)$

27. $(1+x)^n = F(-n, b, b; -x)$, $\quad (1-x)^n = 1 - nxF(1-n, 1, 2; x)$

28. arc tan $x = xF(\frac{1}{2}, 1, \frac{3}{2}; -x^2)$, arc sin $x = xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2)$

29. $\ln(1 + x) = xF(1, 1, 2; -x)$, $\ln \frac{1 + x}{1 - x} = 2xF(\frac{1}{2}, 1, \frac{3}{2}; x^2)$

30. **(Second solution)** Show that for $r_2 = 1 - c$ in Prob. 21 the Frobenius method yields the following solution (where $c \neq 2, 3, 4, \cdots$):

$$y_2(x) = x^{1-c}\left(1 + \frac{(a-c+1)(b-c+1)}{1!(-c+2)}x + \frac{(a-c+1)(a-c+2)(b-c+1)(b-c+2)}{2!(-c+2)(-c+3)}x^2 + \cdots\right) \quad (19)$$

31. Show that in Prob. 30,

$$y_2(x) = x^{1-c}F(a - c + 1, b - c + 1, 2 - c; x).$$

32. Show that if $c \neq 0, \pm 1, \pm 2, \cdots$, the functions (18) and (19) constitute a basis of solutions of (17).

33. Consider the differential equation

$$(t^2 + At + B)\ddot{y} + (Ct + D)\dot{y} + Ky = 0 \quad (20)$$

where A, B, C, D, K are constants, $\dot{y} = dy/dt$, and $t^2 + At + B$ has distinct zeros t_1 and t_2. Show that by introducing the new independent variable

$$x = \frac{t - t_1}{t_2 - t_1}$$

the equation (20) becomes the hypergeometric equation, where the parameters are related by $Ct_1 + D = -c(t_2 - t_1)$, $C = a + b + 1$, $K = ab$.

Solve the following equations in terms of hypergeometric functions.

34. $x(1 - x)y'' + (3 - 5x)y' - 4y = 0$
35. $8x(1 - x)y'' + (4 - 14x)y' - y = 0$
36. $x(1 - x)y'' + (\frac{1}{2} + 2x)y' - 2y = 0$
37. $5x(1 - x)y'' + (4 - x)y' + y = 0$
38. $4x(1 - x)y'' + (1 - 10x)y' - 2y = 0$
39. $4x(1 - x)y'' + y' + 8y = 0$
40. $2x(1 - x)y'' + (7 + 2x)y' - 2y = 0$
41. $4x(1 - x)y'' + (2 + 12x)y' - 15y = 0$
42. $3x(1 - x)y'' + (1 - 5x)y' + y = 0$
43. $4(t^2 - 3t + 2)\ddot{y} - 2\dot{y} + y = 0$
44. $2(t^2 - 5t + 6)\ddot{y} + (2t - 3)\dot{y} - 8y = 0$
45. $3t(1 + t)\ddot{y} + t\dot{y} - y = 0$

Problem Set 5.5:

1. Show that the series in (11) converge for all x.
2. The series (11)–(13) converge very rapidly (why?), so that they are useful in computations. For illustration, find out how many terms of (12) one needs to compute $J_0(1)$ with an error less than 1 unit of the 5th decimal place. (*Hint*. Use the Leibniz test in Appendix 3.) How many terms would you need to compute ln 2 from the Maclaurin series of $\ln(1 + x)$ with the same accuracy?
3. Show that $J_n(x)$ for even n is an even function and for odd n is an odd function.
4. Show that for small $|x|$ we have $J_0(x) \approx 1 - 0.25x^2$. Using this formula, compute $J_0(x)$ for $x = 0.1, 0.2, \cdots, 1.0$ and determine the relative error by comparing with Table A1 in Appendix 5.
5. **(Behavior for large x)** It can be shown that for large x,

$$\begin{aligned} J_{2n}(x) &\approx (-1)^n(\pi x)^{-1/2}(\cos x + \sin x) \\ J_{2n+1}(x) &\approx (-1)^{n+1}(\pi x)^{-1/2}(\cos x - \sin x). \end{aligned} \quad (24)$$

Using (24), sketch $J_0(x)$ for large x, compute approximate values of the first five positive zeros of $J_0(x)$, and compare them with the more accurate values 2.405, 5.520, 8.654, 11.792, 14.931.

6. Using (12) and the Leibniz test in Appendix 3, can you think of an argument why $2 < x_0 < \sqrt{8}$, where $x_0 \approx 2.405$ is the smallest positive zero of $J_0(x)$?
7. Using (24), compute approximate values of the first four positive zeros of $J_1(x)$ and determine the relative error, using the more exact values 3.832, 7.016, 10.173, 13.324.
8. Using (12) and (13), show that $J_0'(x) = -J_1(x)$.
9. Referring to Prob. 8, does Fig. 84 give the impression that $J_1(x) = 0$ when $J_0(x)$ has a horizontal tangent?
10. Using (12) and (13), show that $J_1'(x) = J_0(x) - \frac{1}{x}J_1(x)$.

Differential equations reducible to Bessel's equation

Various differential equations can be reduced to Bessel's equation. To see this, use the indicated substitutions and find a general solution in terms of J_ν and $J_{-\nu}$, or indicate why these functions do not give a general solution. (***More such equations follow in Problem Set 5.7.***)

11. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
12. $x^2y'' + xy' + (x^2 - 16)y = 0$
13. $4x^2y'' + 4xy' + (100x^2 - 9)y = 0$ $(5x = z)$
14. $4x^2y'' + 4xy' + (x - \frac{1}{36})y = 0$ $(\sqrt{x} = z)$
15. $9x^2y'' + 9xy' + (36x^4 - 16)y = 0$ $(x^2 = z)$
16. $xy'' + 2y' + xy = 0$ $(y = u/\sqrt{x})$
17. $xy'' + 5y' + xy = 0$ $(y = u/x^2)$
18. $xy'' - 5y' + xy = 0$ $(y = x^3u)$
19. $81x^2y'' + 27xy' + (9x^{2/3} + 8)y = 0$ $(y = x^{1/3}u,\ x^{1/3} = z)$
20. $x^2y'' + \frac{1}{2}xy' + \frac{1}{16}(x^{1/2} + \frac{15}{16})y = 0$ $(y = x^{1/4}u,\ x^{1/4} = z)$

Problem Set 5.6:

Using (1)–(4), show that

1. $J_0'(x) = -J_1(x)$
2. $J_1'(x) = J_0(x) - x^{-1}J_1(x)$
3. $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$
4. $J_2'(x) = (1 - 4x^{-2})J_1(x) + 2x^{-1}J_0(x)$
5. Using Table A1 in Appendix 5 and (3), compute $J_2(x)$ for the values $x = 0, 0.1, 0.2, \cdots, 1.0$.
6. Compute $J_3(x)$ for the values $x = 2.0, 2.2, 2.4, 2.6, 2.8$ from (3) and Table A1 in Appendix 5.
7. Derive Bessel's equation from (1) and (2).
8. **(Interlacing of zeros)** Using (1), (2), and Rolle's theorem, show that between two consecutive zeros of $J_0(x)$ there is precisely one zero of $J_1(x)$.
9. Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.

Integrals involving Bessel functions can often be evaluated or at least simplified by the use of (1)–(4). Show that

10. $\int x^\nu J_{\nu-1}(x)\,dx = x^\nu J_\nu(x) + c$
11. $\int x^{-\nu} J_{\nu+1}(x)\,dx = -x^{-\nu}J_\nu(x) + c$
12. $\int J_{\nu+1}(x)\,dx = \int J_{\nu-1}(x)\,dx - 2J_\nu(x)$

Using the formulas in Probs. 10–12 and, if necessary, integration by parts, evaluate

13. $\int J_3(x)\,dx$
14. $\int x^3 J_0(x)\,dx$
15. $\int J_5(x)\,dx$
16. Derive the formulas in Example 4 of the text.
17. **(Gamma function)** Using (5) and $\sqrt{\pi} = 1.772\,454$, compute $\Gamma(1.5)$, $\Gamma(2.5)$, and $\Gamma(3.5)$.
18. Compute $\Gamma(4.6)$ from Table A2 in Appendix 5.

Elimination of first derivative

19. Substitute $y(x) = u(x)v(x)$ into $y'' + p(x)y' + q(x)y = 0$ and show that for obtaining a second-order differential equation for u not containing u', we must take

$$v(x) = \exp\left(-\frac{1}{2}\int p(x)\,dx\right).$$

20. Show that for the Bessel equation the substitution in Prob. 19 is $y = ux^{-1/2}$ and gives

$$x^2u'' + (x^2 + \tfrac{1}{4} - \nu^2)u = 0. \qquad (8)$$

Solve this equation with $\nu = \frac{1}{2}$ and compare the result with (6) and (7). Comment.

Problem Set 5.7:

Some further differential equations reducible to Bessel's equation (See also Sec. 5.5.) Using the indicated substitutions, reduce the following equations to Bessel's differential equation and find a general solution in terms of Bessel functions.

1. $x^2y'' + xy' + (x^2 - 4)y = 0$
2. $x^2y'' + xy' + (\lambda^2x^2 - \nu^2)y = 0 \quad (\lambda x = z)$
3. $xy'' + y' + \frac{1}{4}y = 0 \quad (\sqrt{x} = z)$
4. $4x^2y'' + 4xy' + (x - n^2)y = 0 \quad (\sqrt{x} = z)$
5. $x^2y'' + xy' + (4x^4 - \frac{1}{4})y = 0 \quad (x^2 = z)$
6. $xy'' + (1 + 2n)y' + xy = 0 \quad (y = x^{-n}u)$
7. $x^2y'' - 3xy' + 4(x^4 - 3)y = 0 \quad (y = x^2u,\ x^2 = z)$
8. $x^2y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0 \quad (y = x^\nu u,\ x^\nu = z)$
9. $x^2y'' + \frac{1}{4}(x + \frac{3}{4})y = 0 \quad (y = u\sqrt{x},\ \sqrt{x} = z)$
10. $y'' + xy = 0 \quad (y = u\sqrt{x},\ \frac{2}{3}x^{3/2} = z)$
11. $y'' + x^2y = 0 \quad (y = u\sqrt{x},\ \frac{1}{2}x^2 = z)$
12. $y'' + k^2xy = 0 \quad (y = u\sqrt{x},\ \frac{2}{3}kx^{3/2} = z)$
13. $y'' + k^2x^2y = 0 \quad (y = u\sqrt{x},\ \frac{1}{2}kx^2 = z)$
14. $y'' + k^2x^4y = 0 \quad (y = u\sqrt{x},\ \frac{1}{3}kx^3 = z)$
15. Show that for small $x > 0$ we have $Y_0(x) \approx 2\pi^{-1}(\ln \frac{1}{2}x + \gamma)$. Using this formula, compute an approximate value of the smallest positive zero of $Y_0(x)$ and compare it with the more accurate value 0.9.
16. It can be shown that for large x,

$$Y_n(x) \approx \sqrt{2/(\pi x)}\,\sin\left(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right). \qquad (11)$$

Using (11), sketch $Y_0(x)$ and $Y_1(x)$ for $0 < x \leqq 15$. Using (11), compute approximate values of the first three positive zeros of $Y_0(x)$ and compare these values with the more accurate values 0.89, 3.96, and 7.09.

17. Show that the Hankel functions (10) constitute a basis of solutions of Bessel's equation for any ν.

Modified Bessel functions

18. The function $I_\nu(x) = i^{-\nu}J_\nu(ix)$, $i = \sqrt{-1}$, is called the *modified Bessel function of the first kind of order* ν. Show that $I_\nu(x)$ is a solution of the differential equation

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0 \qquad (12)$$

and has the representation

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{x^{2m+\nu}}{2^{2m+\nu}m!\,\Gamma(m + \nu + 1)}. \qquad (13)$$

19. Show that $I_\nu(x)$ is real for all real x (and real ν), $I_\nu(x) \neq 0$ for all real $x \neq 0$, and $I_{-n}(x) = I_n(x)$, where n is any integer.

20. Show that another solution of the differential equation (12) is the so-called *modified Bessel function of the third kind* (sometimes: *of the second kind*)

$$K_\nu(x) = \frac{\pi}{2 \sin \nu\pi}[I_{-\nu}(x) - I_\nu(x)]. \tag{14}$$

Problem Set 5.8:

1. Carry out the details of the proof of Theorem 1 in Cases 3 and 4.

2. **(Normalization of eigenfunctions)** Show that if $y = y_0$ is an eigenfunction of (1), (2) corresponding to some eigenvalue $\lambda = \lambda_0$, then $y = \alpha y_0$ ($\alpha \neq 0$, arbitrary) is an eigenfunction of (1), (2) corresponding to λ_0. (Note that this property can be used to "normalize" eigenfunctions, that is, to obtain eigenfunctions of norm 1.)

Find the eigenvalues and eigenfunctions of the following Sturm–Liouville problems. In Probs. 3–9 also verify orthogonality by direct calculation.

3. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$

4. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$

5. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$

6. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$

7. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$

8. $y'' + \lambda y = 0, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$

9. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$

10. $(xy')' + \lambda x^{-1} y = 0, \quad y(1) = 0, \quad y(e) = 0.$ *Hint.* Set $x = e^t$.

11. $(xy')' + \lambda x^{-1} y = 0, \quad y(1) = 0, \quad y'(e) = 0$

12. $(e^{2x} y')' + e^{2x}(\lambda + 1)y = 0, \quad y(0) = 0, \quad y(\pi) = 0$ *Hint.* Set $y = e^{-x}u$.

13. $(x^{-1}y')' + (\lambda + 1)x^{-3} y = 0, \quad y(1) = 0, \quad y(e) = 0$

14. Show that the eigenvalues of the Sturm–Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$ are obtained as solutions of the equation $\tan k = -k$, where $k = \sqrt{\lambda}$. Show graphically that this equation has infinitely many solutions $k = k_n$ and the eigenfunctions are $y_n = \sin k_n x$ ($k_n \neq 0$). Show that the positive k_n are of the form $k_n = \frac{1}{2}(2n + 1)\pi + \delta_n$, where δ_n is positive and small and $\delta_n \to 0$ as $n \to \infty$. Compute k_0 and k_1 (by Newton's method, Sec. 18.2).

15. Verify by direct calculation that $P_0(x)$, $P_1(x)$, $P_2(x)$ form an orthogonal set on $-1 \leqq x \leqq 1$ (with $p(x) = 1$) and find the corresponding orthonormal set, also by direct calculation.

16. Determine constants $a_0, b_0, \cdots, c_2$ so that $y_0 = a_0$, $y_1 = b_0 + b_1 x$, $y_2 = c_0 + c_1 x + c_2 x^2$ form an orthonormal set on $-1 \leqq x \leqq 1$ (with $p(x) = 1$). Compare the result with that of Prob. 15 and comment.

17. Show that if the functions $y_0(x), y_1(x), \cdots$ form an orthogonal set on an interval $a \leqq x \leqq b$ (with $p(x) = 1$), then the functions $y_0(ct + k), y_1(ct + k), \cdots$, $c > 0$, form an orthogonal set on the interval $(a - k)/c \leqq t \leqq (b - k)/c$.

18. Using Prob. 17, derive the orthogonality of 1, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$, $\cdots$ on $-1 \leqq x \leqq 1$ ($p(x) = 1$) from that in Example 4 of the text.

Verify that the given functions are orthogonal on the given interval with respect to the given $p(x)$ and have the indicated norm.

19. $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 1 - 2x + \frac{1}{2}x^2$, $0 \leqq x < \infty$, $p(x) = e^{-x}$, norm 1

20. $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $-1 \leqq x \leqq 1$, $p(x) = (1 - x^2)^{-1/2}$, $\|T_0\| = \sqrt{\pi}$, $\|T_1\| = \|T_2\| = \sqrt{\pi/2}$. *Hint.* Set $x = \cos\theta$.

Problem Set 5.9:

Legendre polynomials. Represent the following polynomials in terms of Legendre polynomials.

1. $5x^3 + x$

2. $10x^3 - 3x^2 - 5x - 1$

3. $1, x, x^2, x^3$

4. $35x^4 + 15x^3 - 30x^2 - 15x + 3$

In each case, obtain the first few terms of the expansion of $f(x)$ in terms of Legendre polynomials and graph the first three partial sums.

5. $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } \quad 0 < x < 1 \end{cases}$

6. $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } \quad 0 < x < 1 \end{cases}$

7. $f(x) = |x|$ if $-1 < x < 1$

8. $f(x) = e^x$ if $-1 < x < 1$

9. Show that the functions $P_n(\cos\theta)$, $n = 0, 1, \cdots$, form an orthogonal set on the interval $0 \leqq \theta \leqq \pi$ with respect to the weight function $\sin\theta$.

Chebyshev polynomials.[20] The functions

$$T_n(x) = \cos(n \arccos x), \quad U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1 - x^2}} \quad (n = 0, 1, \cdots)$$

are called *Chebyshev polynomials of the first and second kind*, respectively.

10. Show that

$$T_0 = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x.$$

11. Show that the Chebyshev polynomials $T_n(x)$ are orthogonal on the interval $-1 \leqq x \leqq 1$ with respect to the weight function $p(x) = 1/\sqrt{1 - x^2}$. *Hint.* To evaluate the integral, set $\arccos x = \theta$.

12. Show that $T_n(x)$ is a solution of the differential equation

$$(1 - x^2)T_n'' - xT_n' + n^2T_n = 0.$$

Laguerre polynomials.[21] The functions

$$L_0 = 1, \qquad L_n(x) = \frac{e^x}{n!}\frac{d^n(x^ne^{-x})}{dx^n}, \qquad n = 1, 2, \cdots$$

are called *Laguerre polynomials*.

13. Show that

$$L_1(x) = 1 - x, \quad L_2(x) = 1 - 2x + x^2/2, \quad L_3(x) = 1 - 3x + 3x^2/2 - x^3/6.$$

14. Verify by direct integration that L_0, $L_1(x)$, $L_2(x)$ are orthogonal on the positive axis $0 \leqq x < \infty$ with respect to the weight function $p(x) = e^{-x}$.

15. Prove that the set of all Laguerre polynomials is orthogonal on $0 \leqq x < \infty$ with respect to the weight function $p(x) = e^{-x}$.

16. Show that

$$L_n(x) = \sum_{m=0}^{n} \frac{(-1)^m}{m!}\binom{n}{m} x^m = 1 - nx + \frac{n(n-1)}{4}x^2 - + \cdots + \frac{(-1)^n}{n!}x^n.$$

17. $L_n(x)$ satisfies Laguerre's differential equation $xy'' + (1 - x)y' + ny = 0$. Verify this fact for $n = 0, 1, 2, 3$.

Hermite polynomials[22]. The functions

$$He_0 = 1, \qquad He_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}), \qquad n = 1, 2, \cdots$$

are called *Hermite polynomials*.

Remark. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \qquad H_n^*(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}.$$

This differs from our definition, which is preferably used in applications.

18. Show that

$$He_1(x) = x, \; He_2(x) = x^2 - 1, \; He_3(x) = x^3 - 3x, \; He_4(x) = x^4 - 6x^2 + 3.$$

19. Show that the Hermite polynomials are related to the coefficients of the Maclaurin series

$$e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x)t^n$$

by the formula $He_n(x) = n!a_n(x)$. *Hint.* Note that $tx - . \;\; 2 = x^2/2 - (x - t)^2/2$. (The exponential function is called the **generating function** of the He_n.)

20. Show that the Hermite polynomials satisfy the relation

$$He_{n+1}(x) = xHe_n(x) - He_n'(x).$$

21. Differentiating the generating function in Prob. 19 with respect to x, show that

$$He_n'(x) = nHe_{n-1}(x).$$

Using this and the formula in Prob. 20 (with n replaced by $n - 1$), prove that $He_n(x)$ satisfies the differential equation

$$y'' - xy' + ny = 0.$$

22. Using the differential equation in Prob. 21, show that $w = e^{-x^2/4}He_n(x)$ is a solution of **Weber's equation**[23]

$$w'' + (n + \tfrac{1}{2} - \tfrac{1}{4}x^2)w = 0 \qquad (n = 0, 1, \cdots).$$

23. Show that the Hermite polynomials are orthogonal on the x-axis $-\infty < x < \infty$ with respect to the weight function $p(x) = e^{-x^2/2}$.

Bessel functions

24. Sketch $J_0(\lambda_{10}x)$, $J_0(\lambda_{20}x)$, $J_0(\lambda_{30}x)$, and $J_0(\lambda_{40}x)$ for $R = 1$ in the interval $0 \leqq x \leqq 1$. (Use Table A1 in Appendix 5.)

Develop the following functions $f(x)$ $(0 < x < R)$ in a **Fourier–Bessel series**

$$f(x) = a_1J_0(\lambda_{10}x) + a_2J_0(\lambda_{20}x) + a_3J_0(\lambda_{30}x) + \cdots$$

and sketch the first few partial sums.

25. $f(x) = 1$. *Hint.* Use (1), Sec. 5.6.

26. $f(x) = \begin{cases} 1 & \text{if } 0 < x < R/2 \\ 0 & \text{if } R/2 < x < R \end{cases}$

27. $f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } a < x < R \end{cases}$

28. $f(x) = \begin{cases} 0 & \text{if } 0 < x < R/2 \\ k & \text{if } R/2 < x < R \end{cases}$

29. $f(x) = 1 - x^2$ $(R = 1)$. *Hint.* Use (1), Sec. 5.6, and integration by parts.[24]

30. $f(x) = R^2 - x^2$

31. $f(x) = x^2$

32. $f(x) = x^4$

33. Show that $f(x) = x^n$ $(0 < x < 1, n = 0, 1, \cdots)$ can be represented by the Fourier–Bessel series

$$x^n = \frac{2J_n(\alpha_{1n}x)}{\alpha_{1n}J_{n+1}(\alpha_{1n})} + \frac{2J_n(\alpha_{2n}x)}{\alpha_{2n}J_{n+1}(\alpha_{2n})} + \cdots.$$

34. Find a representation of $f(x) = x^n$ $(0 < x < R, n = 0, 1, \cdots)$ similar to that in Prob. 33.

35. Represent $f(x) = x^3$ $(0 < x < 2)$ by a Fourier–Bessel series involving J_3.

Respostas:

PROBLEM SET 5.2, page 208

1. $y = -1 + c_3x^3$ **3.** $y = a_0e^{x^2}$

5. $y = a_0(1 + 3x + 4x^2 + \frac{10}{3}x^3 + 2x^4 + \frac{14}{15}x^5 + \cdots) = a_0(x + 1)e^{2x}$

7. $y = c_0 + c_1x + (\frac{3}{2}c_1 - c_0)x^2 + (\frac{7}{6}c_1 - c_0)x^3 + \cdots$. Setting $c_0 = A + B$ and $c_1 = A + 2B$, we obtain $y = Ae^x + Be^{2x}$. This illustrates the fact that even if the solution of an equation is a known function, the power series method may not yield it immediately in the usual form. Practically, this does not matter because the main interest concerns equations for which the power series solutions define *new* functions.

9. $y = a_1x + a_0(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - \cdots)$. [This is a particular case of Legendre's equation ($n = 1$), which we consider in Sec. 5.3.]

11. $(t + 1)\dot{y} - y = t + 1$, $y = c_0(1 + t) + (1 + t)(t - \frac{1}{2}t^2 + - \cdots)$
$= c_0x + x \ln x \qquad (\dot{y} = dy/dt)$

13. $y = a_0(1 - t^2/2! + t^4/4! - + \cdots) + a_1(t - t^3/3! + - \cdots)$
$= a_0 \cos(x - 1) + a_1 \sin(x - 1)$

15. 3 **17.** ∞ **19.** $\sqrt{|k|}$ **21.** $\sqrt{5/7}$ **23.** 0 **25.** $\sqrt{2}$

29. $\sum_{m=1}^{\infty} \frac{(m + 1)(m + 2)}{(m + 1)^2 + 1} x^m$; 1

PROBLEM SET 5.3, page 213

9. Set $(x^2 - 1)^n = v(x)$. Since $v', v'', \cdots, v^{(n-1)}$ are zero at $x \pm 1$ and $v^{(2n)} = (2n)!$, we obtain from (12)

$$(2^n n!)^2 \int_{-1}^{1} P_n^{\,2}\, dx = \int_{-1}^{1} v^{(n)}v^{(n)}\, dx = [v^{(n-1)}v^{(n)}]|_{-1}^{1} - \int_{-1}^{1} v^{(n-1)}v^{(n+1)}\, dx$$

$$= \cdots = (-1)^n \int_{-1}^{1} vv^{(2n)}\, dx = (-1)^n(2n)! \int_{-1}^{1} (x^2 - 1)^n\, dx$$

$$= 2(2n)! \int_{0}^{1} (1 - x^2)^n\, dx = 2(2n)! \int_{0}^{\pi/2} \sin^{2n+1}\beta\, d\beta$$

$$= 2(2n)! \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n + 1)} = \frac{2}{2n + 1}.$$

$(x = \cos\beta)$.

13. Set $x = -1$ and use the formula for the sum of a geometric series.

15. Set $x = 0$ and use $(1 + u^2)^{-1/2} = \sum \binom{-1/2}{m} u^{2m}$.

PROBLEM SET 5.4, page 223

1. $y_1 = x^{-1}\cos x$, $y_2 = x^{-1}\sin x$

3. $y_1 = \frac{1}{x^2}\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) = \frac{\sinh x}{x^3}$, $y_2 = \frac{\cosh x}{x^3}$

5. $y_1 = (\sin x)/x^3$, $y_2 = (\cos x)/x^3$

9. $y_1 = \frac{1}{x^2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right) = \frac{\text{cox } x}{x^2}$, $y_2 = \frac{\sin x}{x^2}$

11. $y_1 = e^x$, $y_2 = e^x \ln x$

13. $y_1 = x^{-2}\sin x^2$, $y_2 = x^{-2}\cos x^2$ **15.** $y_1 = (x - 1)^2$, $y_2 = (x - 1)^{-2}$

17. $y_1 = x$, $y_2 = x \ln x + x^2$ **19.** $y_1 = \sqrt{x}\, e^x$, $y_2 = (x + 1)e^x$

35. $y = AF(\frac{1}{2}, \frac{1}{4}, \frac{1}{2}; x) + B\sqrt{x}\, F(1, \frac{3}{4}, \frac{3}{2}; x)$

37. $y = AF(\frac{1}{5}, -1, \frac{4}{5}; x) + Bx^{1/5}F(\frac{2}{5}, -\frac{4}{5}, \frac{6}{5}; x)$

39. $y = A(1 - 8x + \frac{32}{5}x^2) + Bx^{3/4}F(\frac{7}{4}, -\frac{5}{4}, \frac{7}{4}; x)$

41. $y = AF(-\frac{3}{2}, -\frac{5}{2}, \frac{1}{2}; x) + Bx^{1/2}(1 + \frac{4}{3}x)$

43. $y = C_1F(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; t - 1) + C_2(t - 1)^{1/2}$

45. $y = c_1F(-1, \frac{1}{3}, \frac{1}{3}; t + 1) + c_2(t + 1)^{2/3}F(-\frac{1}{3}, 1, \frac{5}{3}; t + 1)$

PROBLEM SET 5.5, page 231

1. Use (7b), Sec. 5.2, and $\frac{x^{2m+2}}{2^{2m+2+n}(m + 1)!(n + m + 1)!} \Big/ \frac{x^{2m}}{2^{2m+n}m!(n + m)!} \to 0$

as $m \to \infty$ (x fixed).

5. Approximate values $\frac{3}{4}\pi + k\pi = 2.356, 5.498, 8.639, 11.781, 14.923$.

7. $\frac{1}{4}\pi + k\pi$, $k = 1, 2, 3, 4$; 2.5%, 0.8%, 0.4%, 0.2%

11. $AJ_{1/3}(x) + BJ_{-1/3}(x)$ **13.** $AJ_{3/2}(5x) + BJ_{-3/2}(5x)$

15. $AJ_{2/3}(x^2) + BJ_{-2/3}(x^2)$ **17.** $x^{-2}J_2(x)$; see Theorem 2.

19. $x^{1/3}(AJ_{1/3}(x^{1/3}) + BJ_{-1/3}(x^{1/3}))$

PROBLEM SET 5.6, page 235

5. $J_2(x) = 2x^{-1}J_1(x) - J_0(x)$; see (3) with $\nu = 1$.

7. Start from (2), with ν replaced by $\nu - 1$, and replace $J_{\nu-1}$ by using (1).

9. Let $J_n(x_1) = J_n(x_2) = 0$. Then $x_1^{-n}J_n(x_1) = x_2^{-n}J_n(x_2) = 0$, and $[x^{-n}J_n(x)]' = 0$ somewhere between x_1 and x_2 by Rolle's theorem. Now use (2). Then use (1) with $\nu = n + 1$.

13. $-2J_2(x) - J_0(x) + c$ by (4)

15. $-2J_4 - 2J_2 - J_0 + c$ by (4)

17. 0.886 227, 1.329 340, 3.323 351

PROBLEM SET 5.7, page 240

1. $AJ_2(x) + BY_2(x)$

3. $AJ_0(\sqrt{x}) + BY_0(\sqrt{x})$

5. $AJ_{1/4}(x^2) + BY_{1/4}(x^2)$

7. $x^2[AJ_2(x^2) + BY_2(x^2)]$

9. $\sqrt{x}\,[AJ_{1/2}(\sqrt{x}) + BJ_{-1/2}(\sqrt{x})] = x^{1/4}[\tilde{A}\sin\sqrt{x} + \tilde{B}\cos\sqrt{x}]$

11. $\sqrt{x}\,[AJ_{1/4}(\tfrac{1}{2}x^2) + BY_{1/4}(\tfrac{1}{2}x^2)]$

13. $\sqrt{x}[AJ_{1/4}(\tfrac{1}{2}kx^2) + BY_{1/4}(\tfrac{1}{2}kx^2)]$

15. 1.1

17. Set $H_\nu^{(1)} = kH_\nu^{(2)}$, use (10), obtain a contradiction.

19. For $x \neq 0$ all the terms of the series (13) are real and positive.

PROBLEM SET 5.8, page 248

3. $\lambda = [(2n + 1)\pi/2]^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin(\tfrac{1}{2}(2n + 1)\pi x)$

5. $\lambda = [(2n + 1)\pi/2L]^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin((2n + 1)\pi x/2L)$

7. $\lambda = n^2$, $n = 0, 1, \cdots$; $y_n(x) = \cos nx$

9. $\lambda = (n\pi/L)^2$, $n = 0, 1, \cdots$; $y_n(x) = \cos(n\pi x/L)$

11. $\lambda = ((2n + 1)\pi/2)^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin\left((2n + 1)\dfrac{\pi}{2}\ln|x|\right)$

13. $\lambda = n^2\pi^2$, $n = 1, 2, \cdots$; $y_n(x) = x\sin(n\pi\ln|x|)$; Euler-Cauchy equation

15. $P_0/\sqrt{2}$, $\sqrt{\tfrac{3}{2}}P_1(x)$, $\sqrt{\tfrac{5}{2}}P_2(x)$

17. Set $x = ct + k$.

PROBLEM SET 5.9, page 255

1. $4P_1 + 2P_3$

3. $x^2 = \tfrac{1}{3}P_0 + \tfrac{2}{3}P_2$, $x^3 = \tfrac{3}{5}P_1 + \tfrac{2}{5}P_3$

5. $\tfrac{1}{4}P_0 + \tfrac{1}{2}P_1 + \tfrac{5}{16}P_2 + \cdots$

7. $\tfrac{1}{2}P_0 + \tfrac{5}{8}P_2 + \cdots$

15. Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^kL_n\,dx = 0$ for $k < n$:

$$\int_0^\infty e^{-x}x^kL_n(x)\,dx = \frac{1}{n!}\int_0^\infty x^k\frac{d^n}{dx^n}(x^ne^{-x})\,dx = -\frac{k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^ne^{-x})\,dx$$

$$= \cdots = (-1)^k\frac{k!}{n!}\int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^ne^{-x})\,dx = 0.$$

21. $G_x = \sum a_n'(x)t^n = \sum He_n'(x)t^n/n! = tG = \sum He_{n-1}(x)t^n/(n-1)!$, etc.

23. Write $e^{-x^2/2} = v$, $v^{(n)} = d^nv/dx^n$, etc., integrate by parts, use the formula $He_n' = nHe_{n-1}$ (Prob. 21); then, for $n > m$,

$$\int_{-\infty}^{\infty} vHe_mHe_n\,dx = (-1)^n\int_{-\infty}^{\infty} He_mv^{(n)}\,dx = (-1)^{n-1}\int_{-\infty}^{\infty} He_m'v^{(n-1)}\,dx$$

$$= (-1)^{n-1}m\int_{-\infty}^{\infty} He_{m-1}v^{(n-1)}\,dx = \cdots$$

$$= (-1)^{n-m}m!\int_{-\infty}^{\infty} He_0\,v^{(n-m)}\,dx = 0.$$

25. By (1) in Sec. 5.6, with $\nu = 1$,

$$a_m = \frac{2}{R^2J_1^2(\alpha_{m0})}\int_0^R xJ_0\left(\frac{\alpha_{m0}}{R}x\right)dx = \frac{2}{\alpha_{m0}{}^2J_1{}^2(\alpha_{m0})}\int_0^{\alpha_{m0}} wJ_0(w)\,dw$$

$$= \frac{2}{\alpha_{m0}J_1(\alpha_{m0})},\qquad f = 2\left(\frac{J_0(\lambda_{10}x)}{\alpha_{10}J_1(\alpha_{10})} + \frac{J_0(\lambda_{20}x)}{\alpha_{20}J_1(\alpha_{20})} + \cdots\right).$$

27. $a_m = \dfrac{2akJ_1(\alpha_{m0}a/R)}{\alpha_{m0}RJ_1{}^2(\alpha_{m0})}$

29. $a_m = \dfrac{4J_2(\alpha_{m0})}{\alpha_{m0}{}^2J_1{}^2(\alpha_{m0})}$

31. $a_m = \dfrac{2R^2}{\alpha_{m0}J_1(\alpha_{m0})}\left[1 - \dfrac{2J_2(\alpha_{m0})}{\alpha_{m0}J_1(\alpha_{m0})}\right]$

35. $x^3 = 16\left[\dfrac{J_3(\alpha_{13}x/2)}{\alpha_{13}J_4(\alpha_{13})} + \dfrac{J_3(\alpha_{23}x/2)}{\alpha_{23}J_4(\alpha_{23})} + \cdots\right]$