Problem Set 12.1:

1. **(Powers of the imaginary unit)** Show that

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \cdots$$

(11)

$$\frac{1}{i} = -i, \quad \frac{1}{i^2} = -1, \quad \frac{1}{i^3} = i, \cdots,$$

Let $z_1 = 4 - 5i$ and $z_2 = 2 + 3i$. Find (in the form $x + iy$)

2. $z_1 z_2$ **3.** $(z_1 + z_2)^2$ **4.** $1/z_2$ **5.** z_2/z_1
6. $3z_1 - 6z_2$ **7.** $0.2{z_1}^3$ **8.** $z_1/(z_1 + z_2)$ **9.** $338/{z_2}^2$

Find

10. $\operatorname{Re} \dfrac{1}{1+i}$ **11.** $\operatorname{Im} \dfrac{3+4i}{7-i}$ **12.** $\operatorname{Re} \dfrac{(2-3i)^2}{2+3i}$ **13.** $\operatorname{Im} \dfrac{z}{\bar{z}}$
14. $(0.3 + 0.4i)^4$ **15.** $\operatorname{Re} z^2$, $(\operatorname{Re} z)^2$ **16.** $\operatorname{Im} z^3$, $(\operatorname{Im} z)^3$ **17.** $(1 + i)^8$

18. Show that z is pure imaginary if and only if $\bar{z} = -z$.
19. Verify the formulas in (10) for $z_1 = 31 - 34i$ and $z_2 = 2 - 5i$.
20. If the product of two complex numbers is zero, show that at least one factor must be zero.

Problem Set 12.2:

1. **(Multiplication by i)** Show that multiplication of a complex number by i corresponds to a **counterclockwise rotation** of the corresponding vector through the angle $\pi/2$.

Find

2. $|-0.2i|$ **3.** $|1.5 + 2i|$ **4.** $|z|^4$, $|z^4|$ **5.** $|\cos\theta + i \sin\theta|$
6. $\left|\dfrac{\bar{z}}{z}\right|$ **7.** $\left|\dfrac{5+7i}{7-5i}\right|$ **8.** $\left|\dfrac{z+1}{z-1}\right|$ **9.** $\left|\dfrac{(1+i)^6}{i^3(1+4i)^2}\right|$

Represent in polar form:

10. $2i$, $-2i$ **11.** $1 + i$ **12.** -3 **13.** $6 + 8i$
14. $\dfrac{1+i}{1-i}$ **15.** $\dfrac{i\sqrt{2}}{4+4i}$ **16.** $\dfrac{3\sqrt{2}+2i}{-\sqrt{2}-2i/3}$ **17.** $\dfrac{2+3i}{5+4i}$

Determine the principal value of the arguments of

18. $-6 - 6i$ **19.** $-10 - i$ **20.** $-\pi$ **21.** $2 + 2i$

Represent in the form $x + iy$:

22. $4(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$ **23.** $2\sqrt{2}(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi)$
24. $10(\cos 0.4 + i \sin 0.4)$ **25.** $\cos(-1.8) + i \sin(-1.8)$

Find all values of the following roots and plot them in the complex plane.

26. $\sqrt{i}$ **27.** $\sqrt{-8i}$ **28.** $\sqrt{-7-24i}$ **29.** $\sqrt[8]{1}$
30. $\sqrt[4]{-7+24i}$ **31.** $\sqrt[4]{-1}$ **32.** $\sqrt[5]{-1}$ **33.** $\sqrt[3]{1+i}$

Solve the equations:

34. $z^2 + z + 1 - i = 0$ **35.** $z^2 - (5 + i)z + 8 + i = 0$
36. $z^4 - 3(1 + 2i)z^2 - 8 + 6i = 0$

Problem Set 12.4:

Find $f(3 + i)$, $f(-i)$, $f(-4 + 2i)$ where $f(z)$ equals

1. $z^2 + 2z$ **2.** $1/(1 - z)$ **3.** $1/z^3$

Find the real and imaginary parts of the following functions.

4. $f(z) = z/(1 + z)$ **5.** $f(z) = 2z^3 - 3z$ **6.** $f(z) = z^2 + 4z - 1$

Suppose that z varies in a region R in the z-plane. Find the (precise) region in the w-plane in which the corresponding values of $w = f(z)$ lie, and show the two regions graphically.

7. $f(z) = z^2$, $|z| > 3$ **8.** $f(z) = 1/z$, $\operatorname{Re} z > 0$ **9.** $f(z) = z^3$, $|\arg z| \leq \frac{1}{4}\pi$

In each case, find whether $f(z)$ is continuous at the origin, assuming that $f(0) = 0$ and, for $z \neq 0$, the function $f(z)$ equals

10. $\operatorname{Re} z/|z|$ **11.** $\operatorname{Re}(z^2)/|z^2|$ **12.** $\operatorname{Im} z/(1 + |z|)$

Differentiate

13. $(z^2 + i)^3$ **14.** $(z^2 - 4)/(z^2 + 1)$ **15.** $i/(1 - z)^2$
16. $(z + i)/(z - i)$ **17.** $(iz + 2)/(3z - 6i)$ **18.** $z^2/(z + i)^2$

Find the value of the derivative of

19. $(z + i)/(z - i)$ at $-i$ **20.** $(z^2 - i)^2$ at $3 - 2i$ **21.** $1/z^3$ at $3i$
22. $z^3 - 2z$ at $-i$ **23.** $(1 + i)/z^4$ at 2 **24.** $(2 + iz)^6$ at $2i$

25. Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z.

26. Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic. *Hint.* Use the relation $|z + \Delta z|^2 = (z + \Delta z)(\bar{z} + \overline{\Delta z})$.

27. Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

28. If $\lim_{z \to z_0} f(z)$ exists, show that this limit is unique.

29. If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that

$$\lim_{n \to \infty} f(z_n) = f(a).$$

30. If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

Problem Set 12.5:

Are the following functions analytic? [Use (1) or (7).]

1. $f(z) = z^8$
2. $f(z) = \text{Re}\,(z^2)$
3. $f(z) = e^x(\cos y + i \sin y)$
4. $f(z) = i/z^4$
5. $f(z) = 1/(1 - z)$
6. $f(z) = z - \bar{z}$
7. $f(z) = \ln |z| + i \,\text{Arg}\, z$
8. $f(z) = 1/(1 - z^4)$
9. $f(z) = \text{Arg}\, z$
10. $f(z) = z + 1/z$
11. $f(z) = z^2 - \bar{z}^2$
12. $f(z) = e^x(\sin y - i \cos y)$

Are the following functions harmonic? If so, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

13. $u = xy$
14. $v = xy$
15. $u = x/(x^2 + y^2)$
16. $v = 1/(x^2 + y^2)$
17. $u = x^3 - 3xy^2$
18. $u = \sin x \cosh y$
19. $u = e^x \cos 2y$
20. $v = i \ln |z|$
21. $v = (x^2 - y^2)^2$

Determine a, b, c such that the given functions are harmonic and find a conjugate harmonic.

22. $u = e^{2x} \cos ay$
23. $u = \cos bx \cosh y$
24. $u = \sin x \cosh cy$

25. Show that if u is harmonic and v a conjugate harmonic of u, then u is a conjugate harmonic of $-v$.

26. Show that, in addition to (4) and (5),

$$(11) \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

27. Formulas (4), (5), and (11) are needed from time to time. Familiarize yourself with them by calculating $(z^3)'$ by one of them and verifying that the result is as expected.

28. Show that if $f(z)$ is analytic and $\text{Re}\, f(z)$ is constant, then $f(z)$ is constant.

29. **(Identically vanishing derivative)** Using (4), show that an analytic function whose derivative is identically zero is a constant.

30. Derive the Cauchy–Riemann equations in polar form (7) from (1).

Problem Set 12.6:

1. Using the Cauchy–Riemann equations, show that e^z is analytic for all z.

Compute e^z (in the form $u + iv$) and $|e^z|$ if z equals

2. $3 + \pi i$
3. $1 + i$
4. $2 + 5\pi i$
5. $\sqrt{2} - \frac{1}{2}i$
6. $7\pi i/2$
7. $(1 + i)\pi$
8. $-1 + 1.4i$
9. $-9\pi i/2$

Find the real and imaginary parts of

10. e^{-z^2}
11. e^{z^3}
12. $e^{-\pi z}$
13. e^{-2z}

Write the following expressions in the polar form (6).

14. $1 + i$
15. $\sqrt{i}$, $\sqrt{-i}$
16. $\sqrt[n]{z}$
17. $3 + 4i$

Find all values of z such that

18. e^z is real
19. $|e^{-z}| < 1$
20. $e^{\bar{z}} = \overline{e^z}$
21. $\text{Re}\, e^{2z} = 0$

Find all solutions and plot some of them in the complex plane.

22. $e^{3z} = 3$
23. $e^z = -2$
24. $e^z = -3 + 4i$
25. $e^z = 0$

26. Show that $u = e^{xy} \cos (x^2/2 - y^2/2)$ is harmonic and find a conjugate.

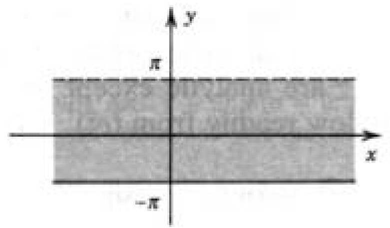

Fig. 300. Fundamental region of the exponential function e^z in the z-plane

27. Find all values of k such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic.

28. Show that $f(z) = e^{\bar{z}}$ is nowhere analytic.

29. It is interesting that $f(z) = e^z$ is *uniquely* determined by the two properties $f(x + i0) = e^x$ and $f'(z) = f(z)$, where f is assumed to be entire. Prove this. *Hint*. Let g be entire with these two properties and show that $(g/f)' = 0$.

30. Prove the statement in Prob. 29, using only the Cauchy–Riemann equations.

Problem Set 12.7:

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.

2. Verify by differentiation that Re $\cos z$ and Im $\sin z$ are harmonic.

Compute (in the form $u + iv$)

3. $\cos(1.7 + 1.5i)$ **4.** $\sin(1.7 + 1.5i)$ **5.** $\sin 10i$

6. $\cos 10i$ **7.** $\sin(\sqrt{2} - 4i)$ **8.** $\cos(\pi + \pi i)$

9. $\cos 3\pi i$ **10.** $\sin(3 + 2i)$ **11.** $\cos(2.1 - 0.2i)$

12. Show that

$$\cosh z = \cosh x \cos y + i \sinh x \sin y,$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

Compute (in the form $u + iv$)

13. $\cosh(-2 + 3i)$ **14.** $\sinh(4 - 3i)$ **15.** $\sinh(2 + i)$

Find all solutions of the following equations.

16. $\cosh z = 0$ **17.** $\cos z = 3i$ **18.** $\sin z = 1000$

19. $\sin z = i \sinh 1$ **20.** $\sin z = \cosh 3$ **21.** $\cosh z = \frac{1}{2}$

22. Find all values of z for which (a) $\cos z$, (b) $\sin z$ has real values.

23. Obtain $\cosh(-1.5 + 1.7i)$ from (15) and the answer to one of the above problems.

24. Find Re $\tan z$ and Im $\tan z$.

25. Prove that $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.

26. Show that $\cos z = \sin(z + \frac{1}{2}\pi)$ and $\sin(\pi - z) = \sin z$, as in real.

27. Show that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$.

28. From (9) and (15) derive the addition rules

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,$$
$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

29. Prove

$$\cos^2 z + \sin^2 z = 1, \qquad \cos^2 z - \sin^2 z = \cos 2z$$
$$\cosh^2 z - \sinh^2 z = 1, \qquad \cosh^2 z + \sinh^2 z = \cosh 2z.$$

30. Show that $|\sinh y| \leqq |\cos z| \leqq \cosh y$ and $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.

Problem Set 13.1:

Find a representation $z = z(t)$ of the straight line segment with endpoints

1. $z = 0$ and $z = 1 + 2i$ **2.** $z = -3 + 2i$ and $z = -4 + 5i$

3. $z = 4 + 2i$ and $z = 3 + 5i$ **4.** $z = 0$ and $z = 5 + 10i$

5. $z = -4i$ and $z = -7 + 38i$ **6.** $z = 1 - i$ and $z = 9 - 5i$

What curves are represented by the following functions?

7. $(1 + 2i)t, \quad 0 \leqq t \leqq 3$ **8.** $3 - it, \quad -1 \leqq t \leqq 1$

9. $1 - i - 2e^{it}, \quad 0 \leqq t \leqq \pi$ **10.** $2 + i + 3e^{it} \quad 0 \leqq t < 2\pi$

11. $t + 3t^2 i, \quad -1 \leqq t \leqq 2$ **12.** $t + 2it^3, \quad -2 \leqq t \leqq 2$

13. $\cos t + 2i \sin t, \quad -\pi < t < \pi$ **14.** $t + t^{-1} i, \quad \frac{1}{2} \leqq t \leqq 5$

Represent the following curves in the form $z = z(t)$.

15. $|z - 3 + 4i| = 4$ **16.** $|z - i| = 2$

17. $y = 1/x$ from $(1, 1)$ to $(3, \frac{1}{3})$ **18.** $y = x^2$ from $(0, 0)$ to $(2, 4)$

19. $x^2 + 4y^2 = 4$ **20.** $4(x - 1)^2 + 9(y + 2)^2 = 36$

Evaluate $\int_C f(z)\,dz$ by the method in Theorem 1 and check the result by Theorem 2:

21. $f(z) = az + b$, C the line segment from $-1 - i$ to $1 + i$

22. $f(z) = e^{2z}$, C the segment in Prob. 1

23. $f(z) = z^3$, C the semicircle $|z| = 2$ from $-2i$ to $2i$ in the right half-plane

24. $f(z) = 5z^2$, C the boundary of the triangle with vertices $0, 1, i$ (clockwise)

Evaluate $\int_C f(z)\,dz$, where

25. $f(z) = 2z^4 - z^{-4}$, C the unit circle (counterclockwise)

26. $f(z) = \text{Re } z$, C the parabola $y = x^2$ from 0 to $1 + i$

27. $f(z) = \text{Im } z$, C the circle $|z| = r$ (counterclockwise)

28. $f(z) = 4z - 3$, C the straight line segment from i to $1 + i$

29. $f(z) = (z - 1)^{-1} + 2(z - 1)^{-2}$, C the circle $|z - 1| = 4$ (clockwise)

30. $f(z) = \sin z$, C the line segment from 0 to i

31. $f(z) = e^{2z}$, C the vertical segment from πi to $2\pi i$
32. $f(z) = z \cos z^2$, C any path from 0 to πi
33. $f(z) = \cosh 3z$, C any path from $\pi i/6$ to 0
34. $f(z) = e^z$, C the boundary of the square with vertices $0, 1, 1 + i, i$ (clockwise)
35. $f(z) = \text{Re}\,(z^2)$, C the square in Prob. 34
36. $f(z) = \text{Im}\,(z^2)$, C the square in Prob. 34
37. $f(z) = \bar{z}$, C the parabola $y = x^2$ from 0 to $1 + i$
38. $f(z) = (z - 1)^{-1} - (z - 1)^{-2}$, C the circle $|z - 1| = \frac{1}{2}$ (clockwise)
39. $f(z) = \sin^2 z$, C the semicircle $|z| = \pi$ from $-\pi i$ to πi in the right half-plane
40. $f(z) = \sec^2 z$, C any path from $\pi i/4$ to $\pi/4$ in the unit disk

41. Evaluate $\int_C \text{Im}\,(z^2)\,dz$ from 0 to $2 + 4i$ along (a) the line segment, (b) the x-axis to 2 and then vertically to $2 + 4i$, (c) the parabola $y = x^2$.

42. Evaluate $\int_C (z^{-5} + z^3)\,dz$ from 1 to -1 along (a) the upper arc of the unit circle, (b) the lower arc of the unit circle.

43. Evaluate $\int_C |z|\,dz$ from A: $z = -i$ to B: $z = i$ along (a) the line segment AB, (b) the unit circle in the left half-plane, (c) the unit circle in the right half-plane.

44. Prove (6) in Sec. 13.1.
45. Verify (6) in Sec. 13.1 for $k_1 f_1 + k_2 f_2 = 3z - z^2$, where C is the upper half of the unit circle from 1 to -1.
46. Verify (7) in Sec. 13.1 for $f(z) = 1/z$, where C is the unit circle, C_1 its upper half, and C_2 its lower half.
47. Verify (8) in Sec. 13.1 for $f(z) = z^2$, where C is the line segment from $-1 - i$ to $1 + i$.

Using the ML-inequality (5), find upper bounds for the following integrals, where C is the line segment from 0 to $3 + 4i$.

48. $\int_C \text{Ln}\,(z + 1)\,dz$ 49. $\int_C e^z\,dz$

50. Find a better bound in Prob. 49 by decomposing C into two arcs.

Problem Set 13.3:

1. Verify Cauchy's integral theorem for the integral of z^2 taken counterclockwise over the boundary of the rectangle with vertices $-1, 1, 1 + i, -1 + i$.
2. Verify Theorem 2 for the integral of $\sin z$ from 0 to $(1 + i)\pi$ (a) over the segment from 0 to $(1 + i)\pi$, (b) over the x-axis to π and then straight up to $(1 + i)\pi$.
3. Verify the result in Example 3.
4. For what contours C will it follow from Cauchy's theorem that

$$\text{(a)} \oint_C \frac{dz}{z} = 0, \qquad \text{(b)} \oint_C \frac{\cos z}{z^6 - z^2}\,dz = 0, \qquad \text{(c)} \oint_C \frac{e^{1/z}}{z^2 + 9}\,dz = 0?$$

5. The integral in Example 4 is zero. Can we conclude from this that it is zero over the contour in Prob. 1?
6. Can we conclude from Example 2 that the integral of $1/(z^2 + 4)$ taken over (a) $|z - 2| = 2$, (b) $|z - 2| = 3$ is zero? Give a reason.

Integrate $f(z)$ counterclockwise over the unit circle and indicate whether Cauchy's theorem may be applied.

7. $f(z) = |z|$
8. $f(z) = e^{z^2}$
9. $f(z) = \text{Im}\,z$
10. $f(z) = 1/(2z - 5)$
11. $f(z) = 1/\bar{z}$
12. $f(z) = 1/(\pi z - 3)$
13. $f(z) = \tan z$
14. $f(z) = \bar{z}$
15. $f(z) = \bar{z}^2$
16. $f(z) = 1/|z|^3$
17. $f(z) = 1/(z^2 + 2)$
18. $f(z) = z^2 \sec z$

Evaluate the following integrals. (*Hint.* If necessary, represent the integrand in terms of partial fractions.)

19. $\oint_C \frac{dz}{z - i}$, C the circle $|z| = 2$ (counterclockwise)
20. $\oint_C \frac{dz}{\sinh z}$, C the circle $|z - \frac{1}{2}\pi i| = 1$ (clockwise)
21. $\oint_C \frac{\cos z}{z}\,dz$, C consists of $|z| = 1$ (counterclockwise) and $|z| = 3$ (clockwise)
22. $\oint_C \frac{2z - 1}{z^2 - z}\,dz$, C the contour in Fig. 323
23. $\oint_C \frac{dz}{z^2 - 1}$, C the contour in Fig. 324
24. $\oint_C \text{Re}\,z\,dz$, C the contour in Fig. 325

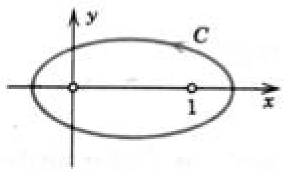

Fig. 323. Problem 22

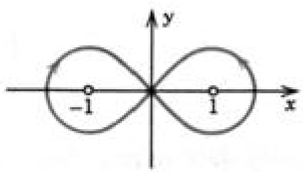

Fig. 324. Problem 23

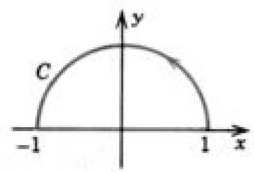

Fig. 325. Problem 24

25. $\oint_C \frac{dz}{z^2+1}$, C: (a) $|z+i| = 1$, (b) $|z-i| = 1$ (counterclockwise)

26. $\oint_C \frac{\sin z}{z+3i}\,dz$, C: $|z - 2 + 3i| = 1$ (counterclockwise)

27. $\oint_C \frac{2z+1}{z^2+z}\,dz$, C: (a) $|z| = \frac{1}{4}$, (b) $|z - \frac{1}{2}| = \frac{1}{4}$, (c) $|z| = 2$ (clockwise)

28. $\oint_C \frac{dz}{1+z^3}$, C: $|z+1| = 1$ (counterclockwise)

29. $\oint_C \frac{3z+1}{z^3-z}\,dz$, C: (a) $|z| = 1/2$, (b) $|z| = 2$ (counterclockwise)

30. $\oint_C \operatorname{Re}(z^2)\,dz$, C the boundary of the triangle with vertices at 0, 2, and $2+i$ (counterclockwise)

Problem Set 13.4:

Evaluate the following integrals.

1. $\int_i^{2+i} z\,dz$
2. $\int_0^{1+i} z^2\,dz$
3. $\int_0^{\pi i} e^z\,dz$
4. $\int_{-\pi i}^{\pi i} \sin^2 z\,dz$
5. $\int_{-\pi i}^{3\pi i} e^{2z}\,dz$
6. $\int_{-i}^{i} \sin z\,dz$
7. $\int_{\pi i/6}^{0} \cosh 3z\,dz$
8. $\int_{-\pi i}^{\pi i} \cos z\,dz$
9. $\int_0^{i} \sinh \pi z\,dz$
10. $\int_{\pi}^{\pi i} \sin 2z\,dz$
11. $\int_{-1}^{1} z \cosh z^2\,dz$
12. $\int_{\pi i}^{0} z \cos z\,dz$
13. $\int_1^{i} z e^{z^2}\,dz$
14. $\int_{1+i}^{1} z^3 e^{z^4}\,dz$
15. $\int_i^{2i} (z^2 - 1)^3\,dz$

Problem Set 13.5:

Integrate $1/(z^4 - 1)$ counterclockwise around the circle

1. $|z - i| = 1$
2. $|z - 1| = 1$
3. $|z + 1| = 1$
4. $|z + 3| = 1$

Integrate $(z^2 - 1)/(z^2 + 1)$ counterclockwise around the circle

5. $|z - 2i| = 2$
6. $|z - i| = 1$
7. $|z| = \frac{1}{2}$
8. $|z + i| = 1$

Integrate the following functions counterclockwise around the unit circle.

9. $(\cos z)/2z$
10. e^z/z
11. $(z+2)/(z-2)$
12. $(e^z - 1)/z$
13. $z^3/(2z - i)$
14. $(\sin z)/2z$
15. $(z - \pi)^{-1} \cos z$
16. $e^{3z}/(3z - i)$

Integrate the given function over the given contour C (counterclockwise or as indicated).

17. $(z^3 - i)/\pi z$, C the circle $|z| = 3$
18. $e^{-3\pi z}/(2z + i)$, C the boundary of the triangle with vertices $-1, 1, -i$
19. $(\tan z)/(z - i)$, C the boundary of the triangle with vertices $-1, 1, 2i$
20. $\cosh(z^2 - \pi i)/(z - \pi i)$, C any ellipse with foci 0 and πi
21. $[\operatorname{Ln}(z - 1)]/(z - 5)$, C the circle $|z - 4| = 2$
22. $[\operatorname{Ln}(z + 1)]/(z^2 + 1)$, C consists of the boundary of the triangle with vertices $1 - \frac{1}{2}i$, $-1 - \frac{1}{2}i$, $2i$ (counterclockwise) and $|z| = \frac{1}{4}$ (clockwise)
23. $e^{z^2}/[(z - 1 - i)z^2]$, C consists of $|z| = 3$ (counterclockwise) and $|z| = 1$ (clockwise)
24. $(\sin z)/(4z^2 - 8iz)$, C consists of the boundaries of the squares with vertices $\pm 3, \pm 3i$ (counterclockwise) and $\pm 1, \pm i$ (clockwise)
25. Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1}\,dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

Problem Set 13.6:

Integrate the following functions counterclockwise around the circle $|z| = 2$. (n in Probs. 10 and 12–15 is a positive integer. a in Probs. 12 and 16 is any number.)

1. $z^4/(z - 3i)^2$
2. $z^2/(z - i)^2$
3. $e^{\pi z}/z^2$
4. $(\cos z)/z^3$
5. $(\cos z)/z^2$
6. $e^{z^2}/(z - 1)^2$
7. $z^3/(z + 1)^3$
8. $(\sin \pi z)/z^3$
9. $(e^z \sin z)/z^2$
10. e^z/z^n
11. e^{z^2}/z^3
12. e^{az}/z^{n+1}
13. $z^n/(z + 1)^{n+1}$
14. $(\sin z)/z^{2n}$
15. $(\cos z)/z^{2n}$
16. $(\sinh az)/z^4$

Integrate $f(z)$ around the contour C (counterclockwise or as indicated).

17. $f(z) = z^{-2} \tan \pi z$, C any ellipse with foci $\pm i$
18. $f(z) = z^{-4} e^{z/2}$, C the circle $|z - 1 - i| = 2$
19. $f(z) = (z - \frac{1}{2}\pi)^{-2} \cot z$, C the boundary of the triangle with vertices $\pm i$ and 2
20. $f(z) = (z - 4)^{-3} \operatorname{Ln} z$, C the circle $|z - 5| = 3$
21. $f(z) = \dfrac{e^{z^2}}{z(z - 2i)^2}$, C consists of the boundary of the square with vertices $\pm 3 \pm 3i$ (counterclockwise) and $|z| = 1$ (clockwise)
22. $f(z) = \dfrac{\operatorname{Ln}(z + 3) + \cos z}{(z + 1)^2}$, C the boundary of the square with vertices $\pm 2, \pm 2i$

23. If $f(z)$ is not a constant and is analytic for all (finite) z, and R and M are any positive real numbers (no matter how large), show that there exist values of z for which $|z| > R$ and $|f(z)| > M$. *Hint*. Use Liouville's theorem.

24. If $f(z)$ is a polynomial of degree $n > 0$ and M an arbitrary positive real number (no matter how large), show that there exists a positive real number R such that $|f(z)| > M$ for all $|z| > R$.

25. Show that $f(z) = e^z$ has the property characterized in Prob. 23, but does not have that characterized in Prob. 24.

26. Prove the **Fundamental theorem of algebra:** If $f(z)$ is a polynomial in z, not a constant, then $f(z) = 0$ for at least one value of z. *Hint*. Assume $f(z) \neq 0$ for all z and apply the result of Prob. 23 to $g = 1/f$.

Problem Set 14.1:

Respostas:

PROBLEM SET 12.1, page 711

3. $32 - 24i$ **5.** $-\frac{7}{41} + \frac{22}{41}i$ **7.** $-47.2 - 23i$ **9.** $-10 - 24i$
11. $31/50$ **13.** $2xy/(x^2 + y^2)$ **15.** $x^2 - y^2, x^2$ **17.** 16

PROBLEM SET 12.2, page 717

3. 2.5 **5.** 1 **7.** 1 **9.** $8/17$
11. $\sqrt{2}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$ **13.** $10(\cos 0.927 + i \sin 0.927)$
15. $\frac{1}{4}(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$ **17.** $0.563(\cos 0.308 + i \sin 0.308)$
19. -3.042 **21.** $\pi/4$ **23.** $-2 + 2i$
25. $-0.227 - 0.974i$ **27.** $\pm(2 - 2i)$
29. $\pm 1, \pm i, \pm(1 \pm i)/\sqrt{2}$ **31.** $\pm(1 \pm i)/\sqrt{2}$
33. $\sqrt[6]{2}\left(\cos \frac{k\pi}{12} + i \sin \frac{k\pi}{12}\right), k = 1, 9, 17$
35. $3 + 2i, 2 - i$ **37.** $|z| = \sqrt{x^2 + y^2} \geqq |x|$, etc.
39. Equation (5) holds when $z_1 + z_2 = 0$. Let $z_1 + z_2 \neq 0$ and $c = a + ib = z_1/(z_1 + z_2)$. By (19) in Prob. 37, $|a| \leqq |c|$, $|a - 1| \leqq |c - 1|$. Thus $|a| + |a - 1| \leqq |c| + |c - 1|$. Clearly $|a| + |a - 1| \geqq 1$. Together we have the inequality below; multiply by $|z_1 + z_2|$ to get (5).

$$1 \leqq |c| + |c - 1| = \left|\frac{z_1}{z_1 + z_2}\right| + \left|\frac{z_2}{z_1 + z_2}\right|$$

PROBLEM SET 12.4, page 725

1. $14 + 8i, -1 - 2i, 4 - 12i$ **3.** $(9 - 13i)/500, -i, (-2 - 11i)/1000$
5. $2(x^3 - 3xy^2) - 3x, 2(3x^2y - y^3) - 3y$
7. $|w| > 9$ **9.** $|\arg w| \leqq 3\pi/4$
11. $\text{Re}\,(z^2)/|z^2| = (x^2 - y^2)/(x^2 + y^2) = 1$ if $y = 0$ and -1 if $x = 0$. *Ans.* No.
13. $6z(z^2 + i)^2$ **15.** $2i/(1 - z)^3$ **17.** 0 **19.** $i/2$
21. $-1/27$ **23.** $-\frac{1}{8}(1 + i)$
25. The quotient in (4) is $\Delta x/\Delta z$, which is 0 if $\Delta x = 0$ but 1 if $\Delta y = 0$, so that it has no limit as $\Delta z \to 0$.
27. Use $\text{Re}\, f(z) = [f(z) + \overline{f(z)}]/2$, $\text{Im}\, f(z) = [f(z) - \overline{f(z)}]/2i$.
29. By continuity, for any $\epsilon > 0$ there is a $\delta > 0$ so that $|f(z) - f(a)| < \epsilon$ when $|z - a| < \delta$. Now $|z_n - a| < \delta$ for all sufficiently large n since $\lim z_n = a$. Thus $|f(z_n) - f(a)| < \epsilon$ for these n.

PROBLEM SET 12.5, page 731

1. Yes **3.** Yes **5.** For $z \neq 1$ **7.** Yes **9.** No
11. No **13.** $f(z) = -iz^2/2$ **15.** $f(z) = 1/z$
17. $f(z) = z^3$ **19.** No **21.** No
23. $b = 1, v = -\sin x \sinh y$
29. $f'(z) = u_x = iv_x = 0$, $u_x = v_x = 0$, hence $v_y = u_y = 0$ by (1), $u = const$, $v = const$, $f = u + iv = const$.

PROBLEM SET 12.6, page 734

3. $1.469 + 2.287i, 2.718$ **5.** $3.610 - 1.972i, 4.113$
7. $-23.141, 23.141$ **9.** $-i, 1$
11. $\exp(x^3 - 3xy^2) \cos(3x^2y - y^3), \exp(x^3 - 3xy^2) \sin(3x^2y - y^3)$
13. $e^{-2x} \cos 2y, -e^{-2x} \sin 2y$ **15.** $e^{\pi i/4}, e^{-3\pi i/4}, e^{-\pi i/4}, e^{3\pi i/4}$
17. $5 \exp(i \arctan \frac{4}{3})$ **19.** $x > 0$
21. $y = (2n + 1)\pi/4, n = 0, \pm 1, \cdots$
23. $z = \ln 2 + (2n + 1)\pi i, n = 0, \pm 1, \cdots$
25. No solutions **27.** $k = 1$
29. $(g/f)' = (g'f - gf')/g^2 = 0$, since $g' = g$, $f' = f$; $g/f = k = const$ by Prob. 29, Sec. 12.5. $g(0) = f(0) = e^0 = 1$ gives $k = 1$, $g = f$.

PROBLEM SET 12.7, page 738

3. $-0.303 - 2.112i$ **5.** $11013i$
7. $26.974 - 4.256i$ **9.** $\cosh 3\pi = 6195.8$
11. $-0.5150 + 0.1738i$ **13.** $-3.725 - 0.512i$
15. $1.960 + 3.166i$ **17.** $\frac{1}{2}(2n + 1)\pi - (-1)^n 1.818i$
19. $\pm 2n\pi + i, \pm(2n + 1)\pi - i$ **21.** $\pm(\pi/3)i \pm 2n\pi i, n = 0, 1, \cdots$
23. Use Prob. 3.

PROBLEM SET 13.1–13.2, page 760

1. $z = (1 + 2i)t, 0 \leqq t \leqq 1$
3. $z = 4 + 2i + (-1 + 3i)t, 0 \leqq t \leqq 1$
5. $z = -4i + (-1 + 6i)t, 0 \leqq t \leqq 7$ **7.** Straight segment from 0 to $3 + 6i$
9. Lower semicircle (radius 2, center $1 - i$)

11. Parabola $y = 3x^2$ from $(-1, 3)$ to $(2, 12)$

13. Ellipse $4x^2 + y^2 = 4$

15. $3 - 4i + 4e^{it}, 0 \leqq t \leqq 2\pi$

17. $t + i/t, 1 \leqq t \leqq 3$

19. $2 \cos t + i \sin t, 0 \leqq t \leqq 2\pi$

21. $2b(1 + i)$ 23. 0 25. 0 27. $-\pi r^2$

29. $-2\pi i$ 31. 0 33. $-i/3$ 35. $-1 - i$

37. $1 + i/3$ 39. $(\pi - \frac{1}{2} \sinh 2\pi)i$

41. $32/3 + 64i/3, 32i, 8 + 128i/5$ 43. $i, 2i, 2i$ 49. $5e^3$

PROBLEM SET 13.3, page 766

5. Yes, by the deformation principle

7. 0, no 9. $-\pi$, no 11. 0, no 13. 0, yes 15. 0, no

17. 0, yes 19. $2\pi i$ 21. 0 23. $2\pi i$ 25. $-\pi, \pi$

27. $-2\pi i, 0, -4\pi i$ 29. $-2\pi i, 0$

PROBLEM SET 13.4, page 770

1. $2 + 2i$ 3. -2 5. 0 7. $-i/3$ 9. $-2/\pi$

11. 0 13. $-\sinh 1$ 15. $-1566i/35$

PROBLEM SET 13.5, page 773

1. $-\pi/2$ 3. $-\pi i/2$ 5. -2π 7. 0 9. πi

11. 0 13. $\pi/8$ 15. 0 17. 2 19. $-2\pi \tanh 1$

21. $2\pi i \operatorname{Ln} 4 = 8.710i$ 23. $\pi e^{-2+2i} = -0.1769 + 0.3866i$

25. Use partial fractions.

PROBLEM SET 13.6, page 778

1. 0 3. $2\pi^2 i$ 5. 0 7. $-6\pi i$ 9. $2\pi i$ 11. 0

13. $2\pi i$ 15. 0 17. $2\pi^2 i$ 19. $-2\pi i$ 21. $\frac{9}{2}\pi e^{-4} i$

Proof. The proof parallels that of Theorem 8.

(a) Let $L = 1 - a^* < 1$. Then by the definition of a limit we have $\sqrt[n]{|z_n|} < q = 1 - \frac{1}{2}a^* < 1$ for all n greater than some (sufficiently large) N^*. Hence $|z_n| < q^n < 1$ for all $n > N^*$. Absolute convergence of the series $z_1 + z_2 + \cdots$ now follows by the comparison with the geometric series.

(b) If $L > 1$, also $\sqrt[n]{|z_n|} > 1$ for all sufficiently large n. Hence $|z_n| > 1$ for those n. Theorem 3 now implies that $z_1 + z_2 + \cdots$ diverges.

(c) Both the *divergent* harmonic series and the *convergent* series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ give $L = 1$. This can be seen from $(\ln n)/n \to 0$ and

$$\sqrt[n]{\frac{1}{n}} = \frac{1}{n^{1/n}} = \frac{1}{e^{(1/n)\ln n}} \to \frac{1}{e^0}, \qquad \sqrt[n]{\frac{1}{n^2}} = \frac{1}{n^{2/n}} = \frac{1}{e^{(2/n)\ln n}} \to \frac{1}{e^0}. \quad ∎$$

EXAMPLE 6 Root test

Is the following series convergent or divergent?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} + 3}(4 - i)^n = \frac{1}{4} - \frac{1}{7}(4 - i) + \frac{1}{19}(4 - i)^2 - + \cdots.$$

Solution. By Theorem 10, the series diverges, since

$$\sqrt[n]{\frac{|(4 - i)^n|}{2^{2n} + 3}} = \frac{|4 - i|}{\sqrt[n]{4^n + 3}} = \frac{\sqrt{17}}{\sqrt[n]{4^n + 3}} \to L = \frac{\sqrt{17}}{4} > 1. \quad ∎$$

This is the end of our discussion of basic concepts and facts on complex series and convergence tests. In the next section we begin our actual work.

Problem Set 14.1

Sequences

Are the following sequences $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find their limit points.

1. $z_n = (-1)^n + 2i$
2. $z_n = e^{in\pi/2}$
3. $z_n = (-1)^n/(n + i)$
4. $z_n = (3 + 4i)^n/n!$
5. $(3i)^n - (1 + i)^n$
6. $z_n = \frac{1}{2}\pi + e^{in\pi/4}/n\pi$
7. $z_n = (-1)^n + i/n$
8. $z_n = n\pi/(1 + 2in)$
9. $z_n = i^n \cos n\pi$

10. **(Uniqueness of limit)** Show that if a sequence converges, its limit is unique.

11. If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ converges with the limit $l + l^*$.

12. Show that under the assumptions of Prob. 11 the sequence $z_1z_1^*, z_2z_2^*, \cdots$ converges with the limit ll^*.

13. Show that a complex sequence $z_1, z_2, \cdots$ is bounded if and only if the two corresponding sequences of the real parts and the imaginary parts are bounded.

Series

14. **(Absolute convergence)** Show that if a series converges absolutely, it is convergent.

Are the following series convergent or divergent?

15. $\sum_{n=0}^{\infty} \frac{(100 + 200i)^n}{n!}$
16. $\sum_{n=0}^{\infty} \frac{n - i}{3n + 2i}$
17. $\sum_{n=1}^{\infty} \frac{i^n}{n}$
18. $\sum_{n=0}^{\infty} n\left(\frac{1 + i}{2}\right)^n$
19. $\sum_{n=0}^{\infty} \left(\frac{8i}{9}\right)^n n^4$
20. $\sum_{n=1}^{\infty} \frac{n^{2n} + i^{2n}}{n!}$
21. $\sum_{n=1}^{\infty} \frac{(2i)^n n!}{n^n}$
22. $\sum_{n=0}^{\infty} \frac{(10 + 7i)^{8n}}{(2n)!}$
23. $\sum_{n=1}^{\infty} \frac{n + 1}{2^n n}$

24. Suppose that $|z_{n+1}/z_n| \leqq q < 1$, so that the series $z_1 + z_2 + \cdots$ converges by the ratio test (Theorem 7). Show that the remainder $R_n = z_{n+1} + z_{n+2} + \cdots$ satisfies $|R_n| \leqq |z_{n+1}|/(1 - q)$. (*Hint.* Use the fact that the ratio test is a comparison of the series $z_1 + z_2 + \cdots$ with the geometric series.)

25. Using Prob. 24, find how many terms suffice for computing the sum s of the series in Prob. 23 with an error not exceeding 0.05 and compute s to this accuracy.

14.2 Power Series

Power series are the most important series in complex analysis, as was mentioned at the beginning of the chapter and as we shall now see in detail.

A **power series** *in powers of* $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \tag{1}$$

where z is a variable, $a_0, a_1, \cdots$ are constants, called the **coefficients** of the series, and z_0 is a constant, called the **center** of the series.

If $z_0 = 0$, we obtain as a particular case a *power series in powers of* z:

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots. \tag{2}$$

Convergence Behavior of Power Series

We have made the definitions in the last section for series of *constant terms*. If the terms of a series are *variable*, say, functions of a variable z (for example, powers of z, as in a power series), they assume definite values if we fix z, and then all those definitions apply. Clearly, for a series of functions of z, the partial sums, the remainders, and the sum will be functions of z. Usually such a series will converge for some z, for instance, throughout

Summary. Power series converge in an open circular disk or some even for every z (or some only at the center, but they are useless); for the radius of convergence, see (5), (6), or Example 6.

Except for the useless ones, power series have sums that are analytic functions (as we show in the next section); this accounts for their importance.

Problem Set 14.2

Find the center and the radius of convergence of the following power series.

1. $\sum_{n=0}^{\infty} (z + 4i)^n$
2. $\sum_{n=1}^{\infty} n\pi^n (z - i)^n$
3. $\sum_{n=0}^{\infty} \left(\frac{\pi}{4}\right)^n z^{2n}$
4. $\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$
5. $\sum_{n=0}^{\infty} \frac{(z - 2i)^n}{5^n}$
6. $\sum_{n=1}^{\infty} \frac{z^{2n}}{n^2}$
7. $\sum_{n=0}^{\infty} \frac{2^{10n}}{n!} (z + i)^n$
8. $\sum_{n=1}^{\infty} n^n (z + 1)^n$
9. $\sum_{n=0}^{\infty} (3z - 2i)^n$
10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$
11. $\sum_{n=0}^{\infty} \frac{i^n n^3}{2^n} z^{2n}$
12. $\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3} (z + \pi i)^n$
13. $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!}$
14. $\sum_{n=1}^{\infty} \frac{(z - 1)^n}{n^n}$
15. $\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n + 1)(n + 2)}$
16. $\sum_{n=1}^{\infty} \frac{n!}{n^n} (z + 2)^n$
17. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1} z^{2n}$
18. $\sum_{n=1}^{\infty} \frac{n^n}{n!} (z - 1)^{2n}$
19. Show that if a power series $\Sigma a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma a_n z^{2n}$ has the radius of convergence $\sqrt{R}$.
20. Does there exist a power series in powers of z that converges at $z = 30 + 10i$ and diverges at $z = 31 - 6i$? (Give a reason.)

14.3 Functions Given by Power Series

The main goal of this section is to show that power series represent analytic functions (Theorem 5). On the way we shall see that power series behave nicely under addition, multiplication, differentiation, and integration, which makes them very useful in complex analysis.

To simplify the formulas in this section, we take $z_0 = 0$ and write

$$\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

This is no restriction, since in a series in powers of $z^* - z_0$ with any center z_0 we can always set $z^* - z_0 = z$ to reduce it to the form (1).

If an arbitrary power series (1) has a nonzero radius of convergence, its sum is a function of z, say, $f(z)$. Then we write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R). \tag{2}$$

We say that $f(z)$ *is* **represented** *by the power series* or that *it is* **developed** *in the power series*. For instance, the geometric series represents the function $f(z) = 1/(1 - z)$ in the interior of the unit circle $|z| = 1$. (See Example 1 in Sec. 14.2.)

Our first goal is to show the **uniqueness** of such a representation; that is, *a function $f(z)$ cannot be represented by two different power series with the same center*. If $f(z)$ can at all be developed in a power series with center z_0, the development is unique. This important fact is frequently used in complex and real analysis. This result (Theorem 2, below) will follow from

Theorem 1 (Continuity of the sum of a power series)
The function $f(z)$ in (2) with $R > 0$ is continuous at $z = 0$.

Proof. By the definition of continuity we must show that

$$\lim_{z \to 0} f(z) = f(0) = a_0,$$

that is, we must show that for a given $\epsilon > 0$ there is a $\delta > 0$ such that $|z| < \delta$ implies $|f(z) - a_0| < \epsilon$. Now (2) converges absolutely for $|z| \leqq r < R$, by Theorem 1 in Sec. 14.2. Hence the series

$$\sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} |a_n| r^n \qquad (r > 0)$$

converges. Let S be its sum. Then, for $0 < |z| \leqq r$,

$$|f(z) - a_0| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \leqq |z| \sum_{n=1}^{\infty} |a_n| \, |z|^{n-1}$$

$$\leqq |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z| S.$$

This is less than ϵ for $|z| < \delta$, where $\delta > 0$ is less than both r and ϵ/S. ∎

From this theorem we can now readily obtain the desired uniqueness theorem (again assuming $z_0 = 0$ without loss of generality):

With Theorem 3 as a tool, we are now ready to establish our main result in this section:

Theorem 5 **(Analytic functions. Their derivatives)**
A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence, by the first statement, each of them represents an analytic function.

Proof. (a) We consider any power series (1) with positive radius of convergence R. Let $f(z)$ be its sum and $f_1(z)$ the sum of its derived series; thus

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{4}$$

We show that $f(z)$ is analytic and has the derivative $f_1(z)$ in the interior of the circle of convergence. We do this by proving that for any fixed z with $|z| < R$ and $\Delta z \to 0$ the difference quotient $[f(z + \Delta z) - f(z)]/\Delta z$ approaches $f_1(z)$. By termwise addition we first have from (4)

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z) = \sum_{n=2}^{\infty} a_n \left[\frac{(z + \Delta z)^n - z^n}{\Delta z} - nz^{n-1}\right]. \tag{5}$$

Note that the summation starts with 2, since the constant term drops out in taking the difference $f(z + \Delta z) - f(z)$, and so does the linear term when we subtract $f_1(z)$ from the difference quotient.

(b) We claim that the series in (5) can be written

$$\sum_{n=2}^{\infty} a_n \Delta z[(z + \Delta z)^{n-2} + 2z(z + \Delta z)^{n-3} + \cdots + (n - 1)z^{n-2}]. \tag{6}$$

The somewhat technical proof of this is given in Appendix 4.

(c) We consider (6). The brackets contain $n - 1$ terms, and the largest coefficient is $n - 1$. Since $(n - 1)^2 < n(n - 1)$, we see that for $|z| \leqq R_0$ and $|z + \Delta z| \leqq R_0$, $R_0 < R$, the absolute value of this series cannot exceed

$$|\Delta z| \sum_{n=2}^{\infty} |a_n| n(n - 1) R_0^{\,n-2}.$$

This series with a_n instead of $|a_n|$ is the second derived series of (2) at $z = R_0$ and converges absolutely by Theorem 3 and Sec. 14.2, Theorem 1. Hence our present series converges. Let $K(R_0)$ be its sum. Then we can write our present result

$$\left|\frac{f(z + \Delta z) - f(z)}{\Delta z} - f_1(z)\right| \leqq |\Delta z| K(R_0).$$

Letting $\Delta z \to 0$ and noting that R_0 ($< R$) is arbitrary, we conclude that $f(z)$ is analytic at any point interior to the circle of convergence and its derivative is represented by the derived series. From this the statements about the higher derivatives follow by induction. ∎

Summary. The results in this section show that power series are about as nice as we could hope for: we can differentiate and integrate them term by term (Theorems 3 and 4). Theorem 5 accounts for the great importance of power series in complex analysis: the sum of such a series (with a positive radius of convergence) is an analytic function and has derivatives of all orders, which thus are analytic functions. But this is only part of the story. In the next section we show that, conversely, *every* given analytic function $f(z)$ can be represented by power series.

Problem Set 14.3

Find the radius of convergence of the following series in two ways, (a) directly by the Cauchy–Hadamard formula (Sec. 14.2), (b) by Theorem 3 or Theorem 4 and a series with simpler coefficients.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z - i)^n$ **2.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n(n+1)} z^n$ **3.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}(z + 1)^{2n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{\pi}\right)^{2n+1}$ **5.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n(n+1)}{7^n} z^{2n}$ **6.** $\displaystyle\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^n}{n(n+1)(n+2)} z^n$ **8.** $\displaystyle\sum_{n=0}^{\infty} \left[\binom{n+k}{n}\right]^{-1} z^{n+k}$ **9.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)(n!)^2} z^{n+1}$

10. In the proof of Theorem 3, we claimed that $\sqrt[n]{n} \to 1$ as $n \to \infty$. Prove this. *Hint.* Set $\sqrt[n]{n} = 1 + c_n$, where $c_n > 0$, and show that $c_n \to 0$ as $n \to \infty$.

11. Show that $(1 - z)^{-2} = \sum_{n=0}^{\infty} (n + 1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

12. Applying Theorem 2 to $(1 + z)^p(1 + z)^q = (1 + z)^{p+q}$ (p and q positive integers), show that

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

13. If $f(z)$ in (1) is even, show that $a_n = 0$ for odd n. (Use Theorem 2.)

14. **(Fibonacci[2] numbers)** The *Fibonacci numbers* are recursively defined by $a_0 = a_1 = 1$, $a_n = a_{n-2} + a_{n-1}$ if $n \geqq 2$. Show that if a power series $a_0 + a_1 z + \cdots$ represents $f(z) = 1/(1 - z - z^2)$, it must have these numbers as coefficients and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

15. Write out the proof on termwise addition and subtraction of power series indicated in the text.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180—1250, Italian

EXAMPLE 3 **Trigonometric and hyperbolic functions**

By substituting (12) into (1) of Sec. 12.7 we obtain

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots \tag{14}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots.$$

When $z = x$ these are the familiar Maclaurin series of the real functions $\cos x$ and $\sin x$. Similarly, by substituting (12) into (11), Sec. 12.7, we obtain

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \tag{15}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

EXAMPLE 4 **Logarithm**

From (9) it follows that

$$\text{Ln}\,(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \qquad (|z| < 1). \tag{16}$$

Replacing z by $-z$ and multiplying both sides by -1, we get

$$-\text{Ln}\,(1 - z) = \text{Ln}\,\frac{1}{1 - z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (|z| < 1). \tag{17}$$

By adding both series we obtain

$$\text{Ln}\,\frac{1 + z}{1 - z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1). \tag{18}$$

In the next section we explain some practical methods of obtaining Taylor series that avoid the cumbersome calculations of the derivatives in (9).

Relation to Last Section

Our discussion in the last section can be nicely related to the present section:

Theorem 2 *Every power series with a nonzero radius of convergence is the Taylor series of the function represented by that power series* (more briefly: *is the Taylor series of its sum*).

Proof. Consider any power series with positive radius of convergence R and call its sum $f(z)$; thus,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots.$$

From Theorem 5 in the last section it follows that

$$f'(z) = a_1 + 2a_2(z - z_0) + \cdots$$

and more generally

$$f^{(n)}(z) = n!a_n + (n + 1)n \cdots 3 \cdot 2a_{n+1}(z - z_0) + \cdots;$$

all these series converge in the disk $|z - z_0| < R$ and represent analytic functions. Hence these functions are continuous at $z = z_0$, by Theorem 1 in the last section. If we set $z = z_0$, we thus obtain

$$f(z_0) = a_0, \qquad f'(z_0) = a_1, \qquad \cdots, \qquad f^{(n)}(z_0) = n!a_n, \qquad \cdots.$$

Since these formulas are identical with those in Taylor's theorem, the proof is complete.

Comment. Comparison with real functions

One surprising property of complex analytic functions is that they have derivatives of all orders, and now we have discovered the other surprising property that they can always be represented by power series of the form (9). This is not true in general for ***real functions;*** there are real functions that have derivatives of all orders but cannot be represented by a power series. (Example: $f(x) = \exp(-1/x^2)$ if $x \neq 0$ and $f(0) = 0$; this function cannot be represented by a Maclaurin series since all its derivatives at 0 are zero.)

Problem Set 14.4

Find the Taylor series of the given function with the given point as center and determine the radius of convergence. (More problems of this kind follow in the next section, after the discussion of practical methods.)

1. e^{-z}, 0 **2.** e^{2z}, $2i$ **3.** $\sin \pi z$, 0
4. $\cos z$, $-\pi/2$ **5.** $\sin z$, $\pi/2$ **6.** $1/z$, 1
7. $1/(1 - z)$, -1 **8.** $1/(1 - z)$, i **9.** $\text{Ln}\, z$, 1
10. $\sinh(z - 2i)$, $2i$ **11.** z^5, -1 **12.** $z^4 - z^2 + 1$, 1
13. $\sin^2 z$, 0 **14.** $\cos^2 z$, 0 **15.** $\cos(z - \pi/2)$, $\pi/2$

Problems 16–26 illustrate how you can obtain properties of functions from their Maclaurin series.

16. Using (12), prove $(e^z)' = e^z$.
17. Derive (14) and (15) from (12). Obtain (16) from Taylor's theorem.
18. Using (14), show that $\cos z$ is even and $\sin z$ is odd.
19. Using (15), show that $\cosh z \neq 0$ for all real $z = x$.
20. Using (14), show that $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$.

EXAMPLE 5 **Use of differential equations**

Find the Maclaurin series of $f(z) = \tan z$.

Solution. We have $f'(z) = \sec^2 z$ and, therefore, since $f(0) = 0$,

$$f'(z) = 1 + f^2(z), \qquad f'(0) = 1.$$

Observing that $f(0) = 0$, we obtain by successive differentiation

$$\begin{aligned} f'' &= 2ff', & f''(0) &= 0, & & \\ f''' &= 2f'^2 + 2ff'', & f'''(0) &= 2, & f'''(0)/3! &= 1/3, \\ f^{(4)} &= 6f'f'' + 2ff''', & f^{(4)}(0) &= 0, & & \\ f^{(5)} &= 6f''^2 + 8f'f''' + 2ff^{(4)}, & f^{(5)}(0) &= 16, & f^{(5)}(0)/5! &= 2/15, \quad \text{etc.} \end{aligned}$$

Hence the result is

$$\text{(3)} \qquad \tan z = z + \frac{1}{3}z^2 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right).$$

EXAMPLE 6 **Undetermined coefficients**

Find the Maclaurin series of $\tan z$ by using those of $\cos z$ and $\sin z$ (Sec. 14.4).

Solution. Since $\tan z$ is odd, the desired expansion will be of the form

$$\tan z = a_1 z + a_3 z^3 + a_5 z^5 + \cdots.$$

Using $\sin z = \tan z \cos z$ and inserting those developments, we obtain

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots = (a_1 z + a_3 z^3 + a_5 z^5 + \cdots)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots\right).$$

Since $\tan z$ is analytic except at $z = \pm\pi/2, \pm 3\pi/2, \cdots$, its Maclaurin series converges in the disk $|z| < \pi/2$, and for these z we may form the Cauchy product of the two series on the right (see Sec. 14.3), that is, multiply the series term by term and arrange the resulting series in powers of z. By Theorem 2 in Sec. 14.3 the coefficient of each power of z is the same on both sides. This yields

$$1 = a_1, \qquad -\frac{1}{3!} = -\frac{a_1}{2!} + a_3, \qquad \frac{1}{5!} = \frac{a_1}{4!} - \frac{a_3}{2!} + a_5, \qquad \text{etc.}$$

Hence $a_1 = 1$, $a_3 = \frac{1}{3}$, $a_5 = \frac{2}{15}$, etc., as before.

Problem Set 14.5

Find the Maclaurin series of the following functions and determine the radius of convergence.

1. $\dfrac{1}{1 + z^4}$
2. $\dfrac{1}{1 - z^5}$
3. $\dfrac{z + 2}{1 - z^2}$
4. $\dfrac{4 - 3z}{(1 - z)^2}$
5. $\sin 2z^2$
6. $\dfrac{1}{(z + 3 - 4i)^2}$
7. $\dfrac{e^{z^4} - 1}{z^3}$
8. $e^{z^2}\displaystyle\int_0^z e^{-t^2}\,dt$
9. $\dfrac{2z^2 + 15z + 34}{(z + 4)^2(z - 2)}$

Find the Taylor series of the given function with the given point as center and determine the radius of convergence.

10. $\dfrac{1}{z}$, 1
11. $\dfrac{1}{z}$, $1 + i$
12. $\dfrac{1}{(z + i)^2}$, $-2i$
13. $z^5 + z^3 - z$, i
14. $(z + i)^3$, $1 - i$
15. $\dfrac{1 + z - \sin(z + 1)}{(z + 1)^3}$, -1
16. e^z, $-\pi i$
17. $\cosh z$, $\pi i/2$
18. $\sin \pi z$, $1/2$

Find the first three nonzero terms of the Taylor series with the given point as center and determine the radius of convergence.

19. $e^{z^2} \sin z^2$, 0
20. $\dfrac{\cos 2z}{1 - 4z^2}$, 0
21. $\tan z$, $\dfrac{\pi}{4}$
22. $e^{z^2}/\cos z$, 0
23. $\cos\left(\dfrac{z}{3 - z}\right)$, 0
24. $\dfrac{4 - 6z}{2z^2 - 3z + 1}$, -1
25. **(Euler numbers)** The Maclaurin series

$$\text{(4)} \qquad \sec z = E_0 - \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 - + \cdots$$

defines the *Euler numbers* E_{2n}. Show that[7] $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$.

26. **(Bernoulli numbers)** The Maclaurin series

$$\text{(5)} \qquad \frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!}z^2 + \frac{B_3}{3!}z^3 + \cdots$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that[7]

$$\text{(6)} \quad B_1 = -\frac{1}{2}, \; B_2 = \frac{1}{6}, \; B_3 = 0, \; B_4 = -\frac{1}{30}, \; B_5 = 0, \; B_6 = \frac{1}{42}, \; \cdots.$$

27. Using (1), (2), Sec. 12.7, and (5), show that

$$\text{(7)} \quad \tan z = \frac{2i}{e^{2iz} - 1} - \frac{4i}{e^{4iz} - 1} - i = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_{2n} z^{2n-1}.$$

28. Developing $1/\sqrt{1 - z^2}$ and integrating, show that

$$\sin^{-1} z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\frac{z^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)\frac{z^7}{7} + \cdots \qquad (|z| < 1).$$

Show that this series represents the principal value of $\sin^{-1} z$ (defined in Prob. 45, Sec. 12.8).

29. Was the radius of convergence in Example 3 to be expected from the form of the given function?
30. Find a Maclaurin series for which the corresponding function has more than one singularity on the circle of convergence.

[7] For tables, see Ref. [1], p. 810, in Appendix 1.

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots,$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$

EXAMPLE 6 Find the Laurent series of $f(z) = 1/(1 - z^2)$ that converges in the annulus $1/4 < |z - 1| < 1/2$ and determine the precise region of convergence.

Solution. The annulus has center 1, so that we must develop

$$f(z) = \frac{-1}{(z-1)(z+1)}$$

in powers of $z - 1$. We calculate

$$\frac{1}{z+1} = \frac{1}{2 + (z-1)} = \frac{1}{2}\,\frac{1}{\left[1 - \left(-\frac{z-1}{2}\right)\right]}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n;$$

this series converges in the disk $|(z - 1)/2| < 1$, that is, $|z - 1| < 2$. Multiplication by $-1/(z - 1)$ now gives the desired series

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} (z-1)^{n-1} = \frac{-1/2}{z-1} + \frac{1}{4} - \frac{1}{8}(z-1) + \frac{1}{16}(z-1)^2 + \cdots.$$

The precise region of convergence is $0 < |z - 1| < 2$; see Fig. 344. We confirm this by noting that $1/(z + 1)$ in $f(z)$ is singular at -1, at distance 2 from the center of the series.

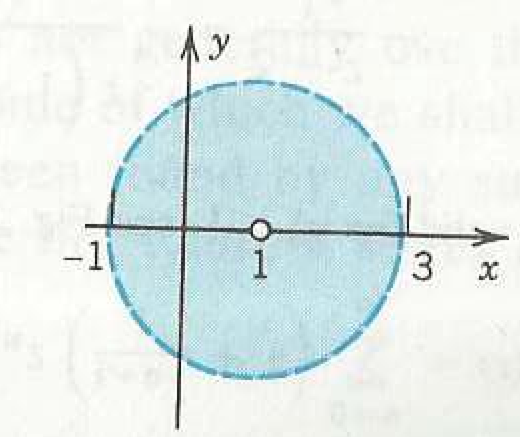

Fig. 344. Region of convergence in Example 6

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

Problem Set 14.7

Expand each of the following functions in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

1. $\dfrac{e^z}{z^2}$ 2. $\dfrac{\sin 4z}{z^4}$ 3. $\dfrac{\cosh 2z}{z}$

4. $\dfrac{1}{z^3(1 - z)}$ 5. $\dfrac{1}{z(1 + z^2)}$ 6. $\dfrac{8 - 2z}{4z - z^3}$

7. $z \cos \dfrac{1}{z}$ 8. $\dfrac{e^{-1/z^2}}{z^5}$ 9. $\dfrac{1}{z^6(1 + z)^2}$

Expand each of the following functions in a Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence.

10. $\dfrac{e^z}{z - 1}$, $z_0 = 1$ 11. $\dfrac{1}{z^2 + 1}$, $z_0 = i$ 12. $z^2 \sinh \dfrac{1}{z}$, $z_0 = 0$

13. $\dfrac{\cos z}{(z - \pi)^3}$, $z_0 = \pi$ 14. $\dfrac{z^4}{(z + 2i)^2}$, $z_0 = -2i$ 15. $\dfrac{z^2 - 4}{z - 1}$, $z_0 = 1$

16. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}$, $z_0 = \dfrac{\pi}{4}$ 17. $\dfrac{1}{(z + i)^2 - (z + i)}$, $z_0 = -i$

18. $\dfrac{1}{1 - z^4}$, $z_0 = -1$

Find the Taylor or Laurent series of $1/(1 - z^2)$ in the region

19. $0 \le |z| < 1$ 20. $|z| > 1$ 21. $0 < |z - 1| < 2$

Using partial fractions, find the Laurent series of $(3z^2 - 6z + 2)/(z^3 - 3z^2 + 2z)$ in the region

22. $0 < |z| < 1$ 23. $1 < |z| < 2$ 24. $|z| > 2$

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise region of convergence.

25. $\dfrac{1}{1 - z^3}$, $z_0 = 0$ 26. $\dfrac{2}{1 - z^2}$, $z_0 = 1$ 27. $\dfrac{z^2}{1 - z^4}$, $z_0 = 0$

28. $\dfrac{1}{z^2}$, $z_0 = i$ 29. $\dfrac{1}{z}$, $z_0 = 1$ 30. $\dfrac{\sinh z}{(z - 1)^2}$, $z_0 = 1$

31. $\dfrac{\sin z}{z + \frac{1}{2}\pi}$, $z_0 = -\dfrac{1}{2}\pi$ 32. $\dfrac{z^3 - 2iz^2}{(z - i)^2}$, $z_0 = i$ 33. $\dfrac{4z - 1}{z^4 - 1}$, $z_0 = 0$

34. Does $\tan(1/z)$ have a Laurent series convergent in a region $0 < |z| < R$?

35. Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

the sphere S, and P^* is the image point of P with respect to this mapping. The complex numbers, first represented in the plane, are now represented by points on S. To each z there corresponds a point on S.

Conversely, each point on S represents a complex number z, except for the point N, which does not correspond to any point in the complex plane. This suggests that we introduce an additional point, called the **point at infinity** and denoted by the symbol ∞ (*infinity*). The complex plane together with the point ∞ is called the **extended complex plane.** The complex plane without that point ∞ is often called the *finite complex plane,* for distinction, or simply the *complex plane,* as before.

Of course, we now let the point $z = \infty$ correspond to N. Then our mapping becomes a one-to-one mapping of the extended complex plane onto S. The sphere S is called the **Riemann number sphere.** The particular mapping we have used is called a **stereographic projection.**

Obviously, the unit circle is mapped onto the "equator" of S. The interior of the unit circle corresponds to the "Southern Hemisphere" and the exterior to the "Northern Hemisphere." Numbers z whose absolute values are large lie close to the North Pole N. The x and y axes (and, more generally, all the straight lines through the origin) are mapped onto "meridians," while circles with center at the origin are mapped onto "parallels." It can be shown that any circle or straight line in the z-plane is mapped onto a circle on S.

Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic or singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

$$g(0) = \lim_{w \to 0} g(w) \tag{4}$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

EXAMPLE 5 Functions analytic or singular at infinity

The function $f(z) = 1/z^2$ is analytic at ∞ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$, and $f(z)$ has a second-order zero at ∞. The function $f(z) = z^3$ is singular at ∞ and has there a pole of third order since $g(w) = f(1/w) = 1/w^3$ has such a pole at $w = 0$. The function e^z has an essential singularity at ∞ since $e^{1/w}$ has such a singularity at $w = 0$. Similarly, $\cos z$ and $\sin z$ have an essential singularity at ∞.

Recall that an **entire function** is one that is analytic everywhere in the (finite) complex plane. Liouville's theorem (Sec. 13.6) tells us that the only *bounded* entire functions are the constants, hence any nonconstant entire function must be unbounded. Hence it has a singularity at ∞, a pole if it is a polynomial or an essential singularity if it is not. The functions just considered are typical in this respect. ∎

A **meromorphic function** is an analytic function whose only singularities in the finite plane are poles.

EXAMPLE 6 Meromorphic functions

Rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are meromorphic functions. ∎

This is the end of Chap. 14 on power series, particularly Taylor series (which play an even greater role here than in calculus), and on Laurent series. Interestingly enough, the latter will provide us with another powerful integration method in the next chapter.

Problem Set 14.8

Singularities. Determine the location and type of the singularities of the following functions, including those at infinity. (In the case of poles also state the order.)

1. $\cot z$
2. $1/(z + a)^4$
3. $z + 1/z$
4. $\frac{3}{z} - \frac{1}{z^2} - \frac{2}{z^3}$
5. $\frac{\cos 4z}{(z^4 - 1)^3}$
6. $\frac{\sin^2 z}{z^4 \cos 2z}$
7. $e^{\pi z}/(z^2 - iz + 2)^2$
8. $e^{1/(z+i)} + z^2$
9. $(e^z - 1 - z)/z^3$
10. $\cosh [1/(z^2 + 1)]$
11. $\tan 1/z$
12. $(\cos z - \sin z)^{-1}$
13. $\cos z - \sin z$
14. $1/\sinh \frac{1}{2}z$
15. $e^{1/(z-1)}/(e^z - 1)$
16. Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

Zeros. Determine the location and order of the zeros of the following functions.

17. $(z^4 - 16)^2$
18. $(z - 16)^8$
19. $z \sin^2 \pi z$
20. $e^z - e^{2z}$
21. $z^{-2} \cos^3 \pi z$
22. $\cosh^2 z$
23. $(3z^2 - 1)/(z^2 - 2iz + 3)^2$
24. $(z^2 - 1)^2(e^{z^2} - 1)$
25. $(1 - \cos z)^2$
26. If $f(z)$ has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$, and the derivative $f'(z)$ has a zero of order $n - 1$ at $z = z_0$ (provided $n > 1$).
27. Prove Theorem 4.
28. If $f_1(z)$ and $f_2(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.
29. Show that the points at which a nonconstant analytic function $f(z)$ assumes a given value k are isolated.

Riemann number sphere. Assuming that we let the image of the x-axis be the meridians 0° and 180°, describe and sketch the images of the following regions on the Riemann number sphere.

30. $|z| \leq 1$
31. First quadrant
32. Second quadrant
33. $|z| > 100$
34. Lower half-plane
35. $\frac{1}{2} \leq |z| \leq 2$

denominator has the limit $q'(z_0)$. Hence our second formula for the residue at a simple pole is

$$\text{(4)} \qquad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

EXAMPLE 4 Residue at a simple pole calculated by formula (4)

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \left[\frac{9z+i}{3z^2+1}\right]_{z=i} = \frac{10i}{-2} = -5i \quad \text{(see Example 3).}$$

EXAMPLE 5 Another application of formula (4)

Find all poles and the corresponding residues of the function

$$f(z) = \frac{\cosh \pi z}{z^4 - 1}.$$

Solution. $p(z) = \cosh \pi z$ is entire, and $q(z) = z^4 - 1$ has simple zeros at $1, i, -1, -i$. Hence $f(z)$ has simple poles at these points (and no further poles). Since $q'(z) = 4z^3$, we see from (4) that the residues equal the values of $(\cosh \pi z)/4z^3$ at those points, that is,

$$\frac{\cosh \pi}{4} \approx 2.8980, \qquad \frac{\cosh \pi i}{4i^3} = \frac{\cos \pi}{-4i} = -\frac{i}{4}, \qquad -\frac{\cosh \pi}{4}, \qquad \frac{\cosh(-\pi i)}{4(-i)^3} = \frac{i}{4}.$$

Formula for the Residue at a Pole of Any Order

Let $f(z)$ be an analytic function that has a pole of any order $m > 1$ at a point $z = z_0$. Then, by the definition of such a pole (Sec. 14.8), the Laurent series of $f(z)$ converging near $z = z_0$ (except at $z = z_0$ itself) is

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

where $b_m \neq 0$. Multiplying both sides by $(z-z_0)^m$, we have

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_2(z-z_0)^{m-2} + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \cdots.$$

We see that the residue b_1 of $f(z)$ at $z = z_0$ is now the coefficient of the power $(z-z_0)^{m-1}$ in the Taylor series of the function

$$g(z) = (z-z_0)^m f(z)$$

on the left, with center $z = z_0$. Thus, by Taylor's theorem (Sec. 14.4),

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0).$$

Hence if $f(z)$ has a pole of mth order at $z = z_0$, the residue is given by

$$\text{(5)} \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right] \right\}.$$

In particular, for a second-order pole ($m = 2$),

$$\text{(5*)} \qquad \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \left\{ [(z-z_0)^2 f(z)]' \right\}.$$

EXAMPLE 6 Residue at a pole of higher order

The function

$$f(z) = \frac{50z}{(z+4)(z-1)^2}$$

has a pole of second order at $z = 1$, and from (5*) we obtain the corresponding residue

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z \to 1} \frac{d}{dz}\left(\frac{50z}{z+4}\right) = 8.$$

EXAMPLE 7 Residues from partial fractions

If $f(z)$ is rational, we can also determine its residues from partial fractions. In Example 6,

$$f(z) = \frac{50z}{(z+4)(z-1)^2} = \frac{-8}{z+4} + \frac{8}{z-1} + \frac{10}{(z-1)^2}.$$

This shows that the residue at $z = 1$ is 8 (as before), and at $z = -4$ (simple pole) it is -8.

Why is this so? Consider $z = 1$. There the Laurent series has the last two fractions as its principal part and the first fraction as the sum of its other part. This first fraction is analytic at $z = 1$, so that it has a Taylor series with center $z = 1$, as it should be. Similarly, at $z = -4$ the first fraction is the principal part of the Laurent series.

EXAMPLE 8 Integration around a second-order pole

Counterclockwise integration of $f(z)$ in Examples 6 and 7 around any simple closed path C such that $z = 1$ is inside C and $z = -4$ is outside C gives (see Example 6 or 7)

$$\oint_C \frac{50z}{(z+4)(z-1)^2}\,dz = 2\pi i \operatorname*{Res}_{z=1} \frac{50z}{(z+4)(z-1)^2} = 2\pi i \cdot 8 = 16\pi i \approx 50.27i.$$

Problem Set 15.1

Find the residues at the singular points of the following functions.

1. $\dfrac{3}{1-z}$
2. $\dfrac{4}{z^3} - \dfrac{1}{z^2}$
3. $\dfrac{\sin z}{z^4}$
4. $\dfrac{z^2+1}{z^2-z}$
5. $\dfrac{z^4}{z^2 - iz + 2}$
6. $\dfrac{e^z}{(z-\pi i)^5}$
7. $\cot z$
8. $\sec z$
9. $2/(z^2-1)^2$

Find the residues at those singular points which lie inside the circle $|z| = 2$.

10. $\frac{z^2}{z^4 - 1}$ 11. $\frac{2z - 3}{z^3 + 3z^2}$ 12. $\frac{z - 23}{z^2 - 4z - 5}$

13. $\frac{3}{(z^4 - 1)^2}$ 14. $\frac{-z^2 - 22z + 8}{z^3 - 5z^2 + 4z}$ 15. $\frac{3z + 6}{(z + 1)(z^2 + 16)}$

Evaluate the following integrals where C is the unit circle (counterclockwise).

16. $\oint_C e^{1/z}\,dz$ 17. $\oint_C \tan z\,dz$ 18. $\oint_C \csc 2z\,dz$

19. $\oint_C \cot z\,dz$ 20. $\oint_C \frac{dz}{\sinh \frac{1}{2}\pi z}$ 21. $\oint_C \frac{z^4 + 6}{z^2 - 2z}\,dz$

22. $\oint_C \frac{\sin \pi z}{z^6}\,dz$ 23. $\oint_C \frac{z^3 + 2}{4z + \pi}\,dz$ 24. $\oint_C \frac{\tanh (z + 1)}{e^z \sin z}\,dz$

25. **Another derivation of (5).** Obtain (5) without using Taylor's theorem, by $m - 1$ differentiations of the formula for $(z - z_0)^m f(z)$.

15.2 Residue Theorem

So far we can evaluate integrals of analytic functions $f(z)$ over closed curves C when $f(z)$ has only *one* singular point inside C. We shall now see that the residue integration method can be extended to the case of several singular points of $f(z)$ inside C. This extension is surprisingly simple, as follows.

Theorem 1 **Residue theorem**

Let $f(z)$ be a function that is analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z), \tag{1}$$

the integral being taken counterclockwise around the path C.

Proof. We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated (Fig. 346). Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz = 0, \tag{2}$$

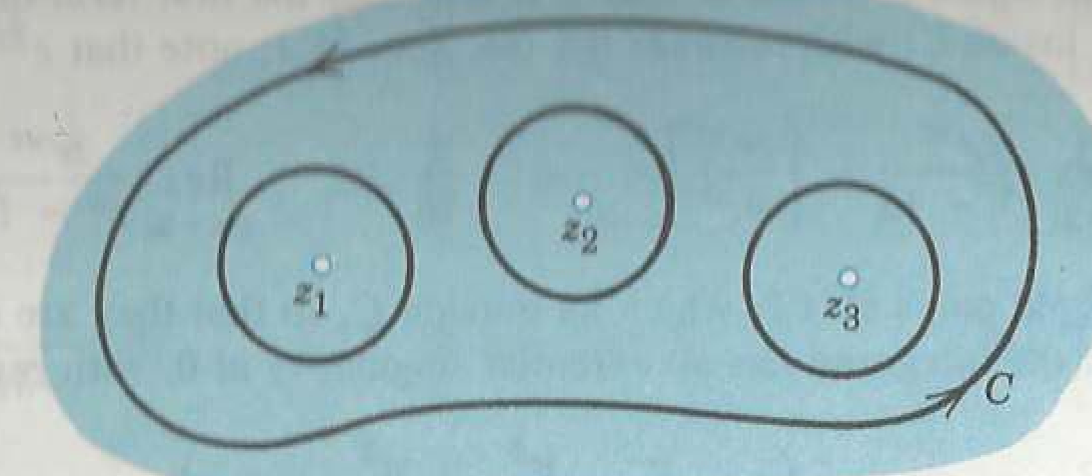

Fig. 346. Residue theorem

the integral along C being taken counterclockwise and the other integrals clockwise (see Sec. 13.3). We now reverse the sense of integration along $C_1, \cdots, C_k$. Then the signs of the values of these integrals change, and we obtain from (2)

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz. \tag{3}$$

All these integrals are now taken counterclockwise. By (1) in the last section,

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z),$$

so that (3) yields (1), and the theorem is proved. ■

This important theorem has various applications in connection with complex and real integrals. We first consider some complex integrals.

EXAMPLE 1 **Integration by the residue theorem**

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4 - 3z}{z^2 - z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3), Sec. 15.1]

$$\operatorname*{Res}_{z=0} \frac{4 - 3z}{z(z - 1)} = \left[\frac{4 - 3z}{z - 1}\right]_{z=0} = -4, \qquad \operatorname*{Res}_{z=1} \frac{4 - 3z}{z(z - 1)} = \left[\frac{4 - 3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4), Sec. 15.1.] *Ans.* (a) $2\pi i(-4 + 1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ■

EXAMPLE 2 **Poles and essential singularities**

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z}\right) dz$$

$z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4), Sec. 15.1; note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=2i} = -\frac{1}{16}, \qquad \operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3}\right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2}\cdot\frac{1}{z} + \cdots.$$

Ans. $2\pi i(-1/16 - 1/16 + \pi^2/2) = \pi(\pi^2 - 1/4)i = 30.221i$ by the residue theorem. ■

EXAMPLE 3 Confirmation of an earlier basic result

Integrate $1/(z - z_0)^m$ (m a positive integer) counterclockwise around any simple closed path C enclosing the point $z = z_0$.

Solution. $1/(z - z_0)^m$ is its own Laurent series with center $z = z_0$ consisting of this one-term principal part, and

$$\operatorname*{Res}_{z=z_0} \frac{1}{z - z_0} = 1,$$

$$\operatorname*{Res}_{z=z_0} \frac{1}{(z - z_0)^m} = 0 \qquad (m = 2, 3, \cdots).$$

In agreement with Example 2, Sec. 13.2, we thus obtain

$$\oint_C \frac{dz}{(z - z_0)^m} = \begin{cases} 2\pi i & \text{if } m = 1 \\ 0 & \text{if } m = 2, 3, \cdots. \end{cases}$$ ■

Problem Set 15.2

Integrate $\dfrac{15z + 9}{z^3 - 9z}$ counterclockwise around the following paths C.

1. $|z| = 1$ **2.** $|z| = 4$ **3.** $|z + 2 + i| = 3$

4. $|z - 3| = 2$ **5.** $|z - \frac{3}{2} + 2i| = 2.4$ **6.** $|z - 1| = 3$

Evaluate the following integrals, where C is any simple closed path such that all the singularities lie inside C (counterclockwise).

7. $\oint_C \dfrac{5z}{z^2 + 4}\,dz$ **8.** $\oint_C \dfrac{z}{1 + 9z^2}\,dz$ **9.** $\oint_C \dfrac{z\cosh \pi z}{z^4 + 13z^2 + 36}\,dz$

10. $\oint_C \dfrac{\sinh z}{2z - i}\,dz$ **11.** $\oint_C \dfrac{z + e^z}{z^3 - z}\,dz$ **12.** $\oint_C \dfrac{z^2 \sin z}{4z^2 - 1}\,dz$

Evaluate the following integrals where C is the unit circle (counterclockwise).

13. $\oint_C \dfrac{z}{z^2 - \frac{1}{4}}\,dz$ **14.** $\oint_C \dfrac{7z}{z^2 + \frac{1}{9}}\,dz$ **15.** $\oint_C \dfrac{dz}{z^2 + 6iz}$

16. $\oint_C \dfrac{e^{-z^2}}{\sin 4z}\,dz$ **17.** $\oint_C \dfrac{e^{-z^2}}{\sin 2z}\,dz$ **18.** $\oint_C \dfrac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)}\,dz$

19. $\oint_C \cot \dfrac{z}{4}\,dz$ **20.** $\oint_C e^z \cot 4z\,dz$ **21.** $\oint_C \dfrac{\sinh z}{4z^2 + 1}\,dz$

22. $\oint_C \dfrac{e^z}{z(z - \pi i/4)^2}\,dz$ **23.** $\oint_C \tan 2\pi z\,dz$ **24.** $\oint_C \dfrac{1 - 4z + 6z^2}{(z^2 + \frac{1}{4})(2 - z)}\,dz$

25. $\oint_C \tan \pi z\,dz$ **26.** $\oint_C \coth z\,dz$ **27.** $\oint_C \dfrac{\tan \pi z}{z^3}\,dz$

28. $\oint_C \dfrac{\cosh z}{z^2 - 3iz}\,dz$ **29.** $\oint_C \dfrac{e^z}{\cos \pi z}\,dz$ **30.** $\oint_C \dfrac{(z + 4)^3}{z^4 + 5z^3 + 6z^2}\,dz$

15.3 Evaluation of Real Integrals

We now show the very surprising fact that the residue theorem also yields a very elegant and simple method for evaluating certain classes of complicated *real* integrals.

Integrals of Rational Functions of cos θ and sin θ

We first consider integrals of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) \end{aligned} \tag{2}$$

and we see that the integrand becomes a rational function of z, say, $f(z)$. As θ ranges from 0 to 2π, the variable z ranges once around the unit circle $|z| = 1$ in the counterclockwise sense. Since $dz/d\theta = ie^{i\theta}$, we have $d\theta = dz/iz$, and the given integral takes the form

$$I = \oint_C f(z)\,\frac{dz}{iz}, \tag{3}$$

the integration being taken counterclockwise around the unit circle.

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z), \tag{7}$$

where we sum over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the upper half-plane.

EXAMPLE 2 An improper integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$

Solution. Indeed, $f(z) = 1/(1 + z^4)$ has four simple poles at the points

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of the these poles lie in the upper half-plane (Fig. 348 on p. 847). From (4) in Sec. 15.1 we find

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1 + z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4} e^{-3\pi i/4} = -\frac{1}{4} e^{\pi i/4},$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1 + z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4} e^{-9\pi i/4} = \frac{1}{4} e^{-\pi i/4}.$$

By (1) in Sec. 12.7 and (7) in the current section,

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{2\pi i}{4}(-e^{\pi i/4} + e^{-\pi i/4}) = \pi \sin \frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1 + x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$ ■

EXAMPLE 3 Another improper integral

Using (7), show that

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{x^4 + 5x^2 + 4}\, dx = \frac{\pi}{6}.$$

Solution. The degree of the denominator is two units higher than that of the numerator, so that our method again applies. Now

$$f(z) = \frac{p(z)}{q(z)} = \frac{z^2 - 1}{z^4 + 5z^2 + 4} = \frac{z^2 - 1}{(z^2 + 4)(z^2 + 1)}$$

has simple poles at $2i$ and i in the upper half-plane (and at $-2i$ and $-i$ in the lower half-plane, which are of no interest here). We calculate the residues from (4), Sec. 15.1, noting that $q'(z) = 4z^3 + 10z$.

$$\operatorname{Res} f(z) = \left[\frac{z^2 - 1}{4z^3 + 10z}\right] = \frac{5}{12i}, \qquad \operatorname{Res} f(z) = \left[\frac{z^2 - 1}{\dots}\right] = \frac{-2}{\dots}$$

Looking back, we realize that the key ideas of our present methods were these. In the first method we mapped the interval of integration on the real axis onto a closed curve in the complex plane (the unit circle). In the second method we attached to an interval on the real axis a semicircle such that we got a closed curve in the complex plane, which we then "blew up." This second method can be applied to further types of integrals, as we show in the next section, the last in this chapter.

Problem Set 15.3

Evaluate the following integrals involving cosine and sine.

1. $\int_0^{2\pi} \frac{d\theta}{1 + \frac{1}{2}\cos\theta}$

2. $\int_0^{\pi} \frac{d\theta}{\pi + \cos\theta}$

3. $\int_0^{2\pi} \frac{d\theta}{37 - 12\cos\theta}$

4. $\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$

5. $\int_0^{2\pi} \frac{d\theta}{5/4 - \sin\theta}$

6. $\int_0^{2\pi} \frac{\cos\theta}{3 + \sin\theta}\, d\theta$

7. $\int_0^{2\pi} \frac{\cos\theta}{17 - 8\cos\theta}\, d\theta$

8. $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$

9. $\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta$ *Hint.* Use $\cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$.

10. $\int_0^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta}\, d\theta$

Evaluate the following improper integrals.

11. $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$

12. $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3}$

13. $\int_0^{\infty} \frac{1 + x^2}{1 + x^4}\, dx$

14. $\int_{-\infty}^{\infty} \frac{x}{x^4 + 1}\, dx$

15. $\int_{-\infty}^{\infty} \frac{dx}{1 + x^6}$

16. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 16}$

17. $\int_{-\infty}^{\infty} \frac{x}{(x^2 - 2x + 2)^2}\, dx$

18. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 9)}$

19. $\int_{-\infty}^{\infty} \frac{dx}{(4 + x^2)^2}$

20. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$

15.4 Further Types of Real Integrals

There are further classes of real integrals that can be evaluated by applying the residue theorem to suitable complex integrals. In applications such integrals may arise in connection with integral transforms or representations of special functions. In the present section we consider two such classes of integrals. One is important in problems involving the Fourier integral rep-

$$\text{pr. v.} \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^{1} \frac{dx}{x^3} \right] = 0;$$

the principal value exists, although the integral itself has no meaning. The whole situation is quite similar to that in the second part of Sec. 15.3.

To evaluate an improper integral whose integrand has poles on the real axis, we use a path that avoids these singularities by following small semicircles with centers at the singular points; this method may be illustrated by the following example.

EXAMPLE 2 **Integrand having a pole on the real axis. Sine integral**

Show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(This is the limit of the sine integral Si(x) as $x \to \infty$; see Sec. 10.9.)

Solution. (a) We do not consider $(\sin z)/z$ because this function does not behave suitably at infinity. We consider e^{iz}/z, which has a simple pole at $z = 0$, and integrate around the contour in Fig. 349. Since e^{iz}/z is analytic inside and on C, Cauchy's integral theorem gives

$$\oint_C \frac{e^{iz}}{z}\, dz = 0. \qquad (7)$$

(b) We prove that the value of the integral over the large semicircle C_1 approaches zero as R approaches infinity. Setting $z = Re^{i\theta}$, we have $dz = iRe^{i\theta}\, d\theta$, $dz/z = i\, d\theta$ and therefore

$$\left| \int_{C_1} \frac{e^{iz}}{z}\, dz \right| = \left| \int_0^\pi e^{iz}\, i\, d\theta \right| \leqq \int_0^\pi |e^{iz}|\, d\theta \qquad (z = Re^{i\theta}).$$

In the integrand on the right,

$$|e^{iz}| = |e^{iR(\cos\theta + i \sin\theta)}| = |e^{iR\cos\theta}|\,|e^{-R\sin\theta}| = e^{-R\sin\theta}.$$

We insert this into the integral and use $\sin(\pi - \theta) = \sin\theta$ to get an integral from 0 to $\pi/2$:

$$\int_0^\pi |e^{iz}|\, d\theta = \int_0^\pi e^{-R\sin\theta}\, d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta.$$

Now Fig. 350 shows that $\sin\theta \geqq 2\theta/\pi$ if $0 \leqq \theta \leqq \pi/2$. Hence $-\sin\theta \leqq -2\theta/\pi$. From this and by integration,

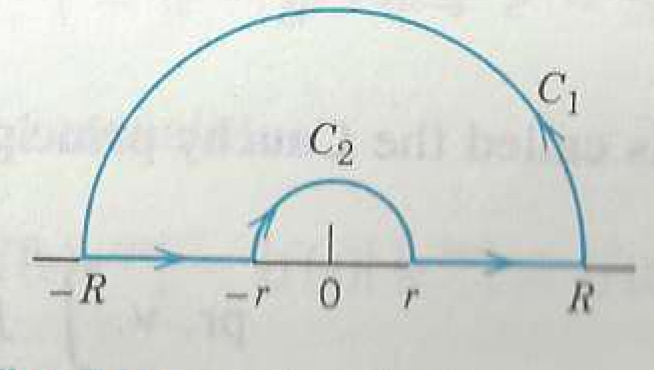

Fig. 349. Contour in Example 2

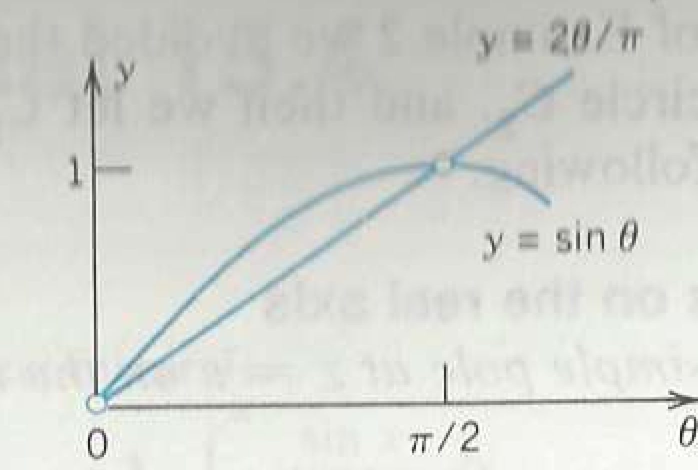

Fig. 350. Inequality in Example 2

$$2\int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \leqq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\, d\theta = \frac{\pi}{R}(1 - e^{-R}) \to 0 \quad \text{as} \quad R \to \infty.$$

Hence the value of the integral over C_1 approaches 0 as $R \to \infty$.

(c) For the integral over the small semicircle C_2 in Fig. 349 we have

$$\int_{C_2} \frac{e^{iz}}{z}\, dz = \int_{C_2} \frac{dz}{z} + \int_{C_2} \frac{e^{iz} - 1}{z}\, dz.$$

The first integral on the right equals $-\pi i$. The integrand of the second integral is analytic and thus bounded, say, less than some constant M in absolute value for all z on C_2 and between C_2 and the x-axis. Hence by the ML-inequality (Sec. 13.2), the absolute value of this integral cannot exceed $M\pi r$. This approaches 0 as $r \to 0$. Because of part (b), from (7) we thus obtain

$$\int_C \frac{e^{iz}}{z}\, dz = \text{pr. v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx + \lim_{r \to 0} \int_{C_2} \frac{e^{iz}}{z}\, dz = \text{pr. v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx - \pi i = 0.$$

Hence this principal value equals πi; its real part is 0 and its imaginary part is

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi. \qquad (8)$$

(d) Now the integrand in (8) is not singular at $x = 0$. Furthermore, since for positive x the function $1/x$ decreases, the areas under the curve of the integrand between two consecutive positive zeros decrease in a monotone fashion, that is, the absolute values of the integrals

$$I_n = \int_{n\pi}^{n\pi + \pi} \frac{\sin x}{x}\, dx \qquad (n = 0, 1, \cdots)$$

form a monotone decreasing sequence $|I_1|, |I_2|, \cdots$, and $I_n \to 0$ as $n \to \infty$. Since these integrals have alternating sign (why?), it follows from the Leibniz test (in Appendix 3) that the infinite series $I_0 + I_1 + I_2 + \cdots$ converges. Clearly, the sum of the series is the integral

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{b \to \infty} \int_0^b \frac{\sin x}{x}\, dx$$

which therefore exists. Similarly, the integral from 0 to $-\infty$ exists. Hence we need not take the principal value in (8), and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

Since the integrand is an even function, the desired result follows. ∎

In part (c) of Example 2 we avoided the simple pole by integrating along a small semicircle C_2, and then we let C_2 shrink to a point. This process suggests the following.

Theorem 1 **Simple poles on the real axis**

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 351)

$$\lim_{r \to 0} \int_{C_2} f(z)\, dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

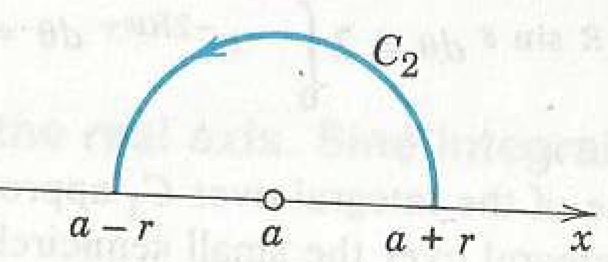

Fig. 351. Theorem 1

Proof. By the definition of a simple pole (Sec. 14.8) the integrand $f(z)$ has at $z = a$ the Laurent series

$$f(z) = \frac{b_1}{z-a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z)$$

where $g(z)$ is analytic on the semicircle of integration (Fig. 351)

$$C_2\colon\ z = a + re^{i\theta}, \qquad 0 \leqq \theta \leqq \pi,$$

and for all z between C_2 and the x-axis. By integration,

$$\int_{C_2} f(z)\, dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}}\, ire^{i\theta}\, d\theta + \int_{C_2} g(z)\, dz.$$

The first integral on the right equals $b_1 \pi i$. The second cannot exceed $M\pi r$ in absolute value, by the ML-inequality (Sec. 13.2), and $M\pi r \to 0$ as $r \to 0$. ∎

We may combine this theorem with (7) of Sec. 15.3 or (3) in this section. Thus [see (7), Sec. 15.3],

(9) $$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

(summation over all poles in the upper half-plane in the first sum, and on the x-axis in the second), valid for rational $f(x) = p(x)/q(x)$ with degree $q \geqq$ degree $p + 2$, having simple poles on the x-axis.

Problem Set 15.4

1. Derive (3) from (2).

Evaluate the following real integrals.

2. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1}\, dx$ **3.** $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^4}\, dx$ **4.** $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+x+1}\, dx$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+x+1}\, dx$ **6.** $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1}\, dx$ **7.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx$ **9.** $\displaystyle\int_{-\infty}^{\infty} \frac{\sin nx}{1+x^4}\, dx$ **10.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4+5x^2+4}\, dx$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+4}\, dx$ **12.** $\displaystyle\int_{\infty}^{\infty} \frac{\cos 2x}{(x^2+4)^2}\, dx$ **13.** $\displaystyle\int_{0}^{\infty} \frac{\cos 2x}{4x^4+13x^2+9}\, dx$

14. Integrating e^{-z^2} around the boundary of the rectangle with vertices $-a$, a, $a+ib$, $-a+ib$, letting $a \to \infty$, and using

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}, \quad \text{show that} \quad \int_0^{\infty} e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2}\, e^{-b^2}.$$

Poles on the real axis. Find the Cauchy principal value of the following integrals.

15. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-ix}$ **16.** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+1)(x^2+2)}$ **17.** $\displaystyle\int_{-\infty}^{\infty} \frac{x}{8-x^3}\, dx$

18. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-2ix}$ **19.** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x-1)}$ **20.** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4-1}$

21. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \frac{1}{4}\pi x}{2x-x^2}\, dx$ **22.** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos \frac{1}{2}\pi x}{x^2-1}\, dx$ **23.** $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{x-x^5}\, dx$

24. Show that the result in Example 2 remains the same if we replace the upper semicircle C_2 by the corresponding lower semicircle.

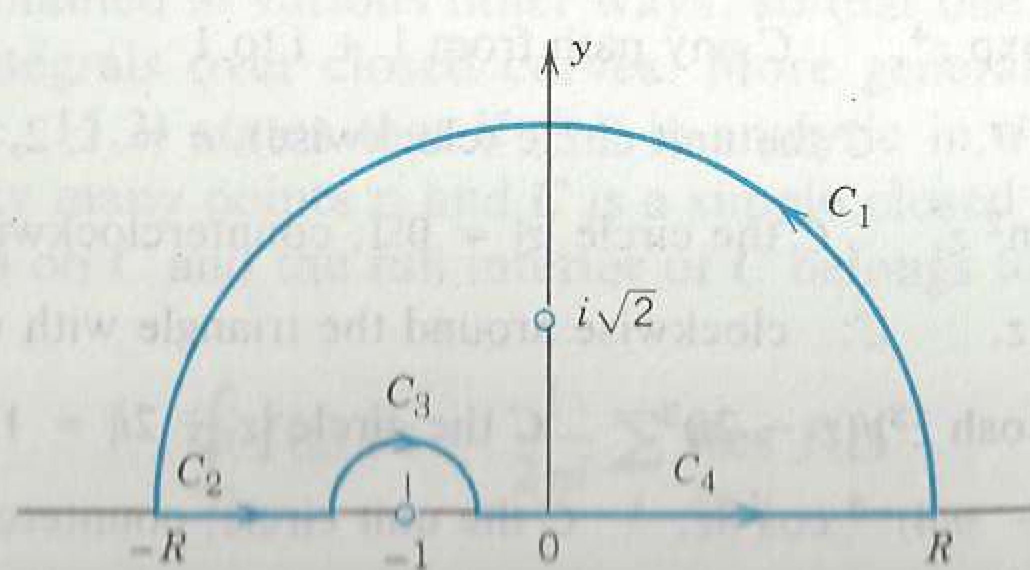

Fig. 352. Problem 16

PROBLEM SET 13.5, page 773

1. $-\pi/2$ 3. $-\pi i/2$ 5. -2π 7. 0 9. πi
11. 0 13. $\pi/8$ 15. 0 17. 2 19. $-2\pi \tanh 1$
21. $2\pi i \operatorname{Ln} 4 = 8.710i$ 23. $\pi e^{-2+2i} = -0.1769 + 0.3866i$
25. Use partial fractions.

PROBLEM SET 13.6, page 778

1. 0 3. $2\pi^2 i$ 5. 0 7. $-6\pi i$ 9. $2\pi i$ 11. 0
13. $2\pi i$ 15. 0 17. $2\pi^2 i$ 19. $-2\pi i$ 21. $\frac{9}{2}\pi e^{-4}i$

CHAPTER 13 (REVIEW QUESTIONS AND PROBLEMS), page 778

17. $-2\pi i$ 19. $2i$ 21. 0 23. $2\pi i$
25. $-2\pi i$ 27. $i \sin 1$ 29. 0 31. $2 \sin 1$
33. $-\frac{1}{2}\sin \pi^2$ 35. $-\pi/2$ 37. $-5 + 8\pi i$ 39. $2\pi^2 i$

PROBLEM SET 14.1, page 790

1. Yes, no, $\pm 1 + 2i$ 3. Yes, yes, 0 5. No, no, none
7. Yes, no, ± 1 9. Yes, no, ± 1, $\pm i$ 15. Convergent
17. Convergent 19. Convergent 21. Divergent
23. Convergent
25. $\dfrac{(n+2)n}{2(n+1)^2} < \dfrac{1}{2}$, $|R_n| \leqq \dfrac{|w_{n+1}|}{1-q} = \dfrac{n+2}{2^n(n+1)} < 0.05$, $n = 5$, $s \approx 1.657$

PROBLEM SET 14.2, page 796

1. $-4i$, 1 3. 0, $2/\sqrt{\pi}$ 5. $2i$, 5 7. $-i$, ∞
9. $2i/3$, $1/3$ 11. 0, $\sqrt{2}$ 13. 0, ∞ 15. 0, 1
17. 0, $\sqrt{2/3}$ 19. $\Sigma a_n z^{2n} = \Sigma a_n (z^2)^n$, $|z^2| < R$, etc.

PROBLEM SET 14.3, page 801

1. 2 3. $\sqrt{5}$ 5. $\sqrt{7/2}$ 7. 1/6 9. 1/4

PROBLEM SET 14.4, page 807

1. $1 - z + z^2/2! - z^3/3! + - \cdots$, $R = \infty$
3. $\pi z - \pi^3 z^3/3! + \pi^5 z^5/5! - + \cdots$, $R = \infty$
5. $1 - (z - \frac{1}{2}\pi)^2/2! + (z - \frac{1}{2}\pi)^4/4! - + \cdots$, $R = \infty$
7. $\frac{1}{2} + \frac{1}{4}(z+1) + \frac{1}{8}(z+1)^2 + \frac{1}{16}(z+1)^3 + \cdots$, $R = 2$
9. $(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - + \cdots$, $R = 1$
11. $-1 + 5(z+1) - 10(z+1)^2 + 10(z+1)^3 - 5(z+1)^4 + (z+1)^5$
13. $\sin^2 z = \frac{1}{2} - \frac{1}{2}\cos 2z = z^2 - 2^3 z^4/4! + 2^5 z^6/6! - + \cdots$, $R = \infty$
15. $1 - (z - \frac{1}{2}\pi)^2/2! + (z - \frac{1}{2}\pi)^4/4! - + \cdots$, $R = \infty$

27. $(2/\sqrt{\pi})(z - z^3/3 + z^5/2!5 - z^7/3!7 + \cdots)$, $R = \infty$
29. $z^3/1!3 - z^7/3!7 + z^{11}/5!11 - + \cdots$, $R = \infty$

PROBLEM SET 14.5, page 810

1. $1 - z^4 + z^8 - z^{12} + - \cdots$, $R = 1$
3. $2 + z + 2z^2 + z^3 + 2z^4 + \cdots$, $R = 1$
5. $2z^2 - 2^3 z^6/3! + 2^5 z^{10}/5! - + \cdots$, $R = \infty$
7. $z + z^5/2! + z^9/3! + \cdots$, $R = \infty$
9. $2/(z-2) - 1/(z+4)^2 = -17/16 - (15/32)z - (67/256)z^2 + \cdots$, $R = 2$
11. $(1-i)/2 - [(1-i)^2/4](z - 1 - i) + [(1-i)^3/8](z - 1 - i)^2 - + \cdots$, $R = \sqrt{2}$
13. $-i + (z-i) - 7i(z-i)^2 - 9(z-i)^3 + 5i(z-i)^4 + (z-i)^5$
15. $1/3! - (z+1)^2/5! + (z+1)^4/7! - + \cdots$, $R = \infty$
17. $i(z - \frac{1}{2}\pi i) + i(z - \frac{1}{2}\pi i)^3/3! + i(z - \frac{1}{2}\pi i)^5/5! + \cdots$, $R = \infty$
19. $z^2 + z^4 + z^6/3 + \cdots$, $R = \infty$
21. $1 + 2(z - \frac{1}{4}\pi) + 2(z - \frac{1}{4}\pi)^2 + \frac{8}{3}(z - \frac{1}{4}\pi)^3 + \cdots$, $R = \frac{1}{4}\pi$
23. $1 - z^2/18 - z^3/27 + \cdots$, $R = 3$

PROBLEM SET 14.6, page 819

1. Use Theorem 1. 3. $R = 1/\sqrt{\pi} > 0.56$
5. $|\sin n|z|| \leqq 1$; $\Sigma\, 1/n^2$ converges. 7. $|z^n| \leqq 1$; $\Sigma\, n/(n^3 + |z|) \leqq \Sigma\, 1/n^2$
9. $|\tanh^n x| \leqq 1$, $1/n(n+1) < 1/n^2$ 11. $|z + 2i| \leqq \sqrt{3} - \delta$, $\delta > 0$
13. $|z| \leqq 4 - \delta$, $\delta > 0$ 15. Nowhere
17. Convergence follows from the comparison test (Sec. 14.1). Let $R_n(z)$ and R_n^* be the remainders of (1) and (5), respectively. Since (5) converges, for given $\epsilon > 0$ we can find an $N(\epsilon)$ such that $R_n^* < \epsilon$ for all $n > N(\epsilon)$. Since $|f_n(z)| \leqq M_n$ for all z in the region G, we also have $|R_n(z)| \leqq R_n^*$ and therefore $|R_n(z)| < \epsilon$ for all $n > N(\epsilon)$ and all z in the region G. This proves that the convergence of (1) in G is uniform.
19. No. Why? 21. $n = 7, 10, 16, 27, 65$

PROBLEM SET 14.7, page 827

1. $\displaystyle\sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots$, $R = \infty$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} z^{2n-1} = \frac{1}{z} + 2z + \frac{2}{3}z^3 + \cdots$, $R = \infty$
5. $\displaystyle\sum_{n=0}^{\infty} (-1)^n z^{2n-1} = \frac{1}{z} - z + z^3 - z^5 + - \cdots$, $R = 1$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{1-2n} = z - \frac{1}{2z} + \frac{1}{24z^3} - + \cdots$, $R = \infty$
9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n z^{n-7} = \frac{1}{z^6} - \frac{2}{z^5} + \frac{3}{z^4} - \frac{4}{z^3} + - \cdots$, $R = 1$

11. $-\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^{n+1}(z-i)^{n-1} = -\frac{i/2}{z-i} + \frac{1}{4} + \frac{i}{8}(z-i) + \cdots, \quad R = 2$

13. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}(z-\pi)^{2n-3} = -\frac{1}{(z-\pi)^3} + \frac{1}{2!(z-\pi)} - \frac{1}{4!}(z-\pi) + - \cdots, \quad R = \infty$

15. $(z-1) + 2 - 3/(z-1)$

17. $-\sum_{n=0}^{\infty}(z+i)^{n-1} = -(z+i)^{-1} - 1 - (z+i) - \cdots, \quad R = 1$

19. $\sum_{n=0}^{\infty} z^{2n}$

21. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+1}}(z-1)^{n-1}$

23. $\frac{1}{z} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

25. $\sum_{n=0}^{\infty} z^{3n}, \quad |z| < 1; \quad -\sum_{n=0}^{\infty}\frac{1}{z^{3n+3}}, \quad |z| > 1$

27. $\sum_{n=0}^{\infty} z^{4n+2}, \quad |z| < 1, \quad -\sum_{n=0}^{\infty}\frac{1}{z^{4n+2}}, \quad |z| > 1$

29. $\sum_{n=0}^{\infty}(-1)^n(z-1)^n, \quad 0 < |z-1| < 1; \quad \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^{n+1}}, \quad |z-1| > 1$

31. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z+\frac{1}{2}\pi)^{2n-1}}{(2n)!}, \quad |z+\frac{1}{2}\pi| > 0$

33. $(1-4z)\sum_{n=0}^{\infty} z^{4n}, \quad |z| < 1, \quad \left(\frac{4}{z^3} - \frac{1}{z^4}\right)\sum_{n=0}^{\infty}\frac{1}{z^{4n}}, \quad |z| > 1$

35. Let $\sum_{-\infty}^{\infty} a_n(z-z_0)^n$ and $\sum_{-\infty}^{\infty} c_n(z-z_0)^n$ be two Laurent series of the same function $f(z)$ in the same annulus. We multiply both series by $(z-z_0)^{-k-1}$ and integrate along a circle with center at z_0 in the interior of the annulus. Since the series converge uniformly, we may integrate term by term. This yields $2\pi i a_k = 2\pi i c_k$. Thus, $a_k = c_k$ for all $k = 0, \pm 1, \cdots$.

PROBLEM SET 14.8, page 833

1. $0, \pm\pi, \pm 2\pi, \cdots$ (simple poles), ∞ (essential singularity)
3. $0, \infty$ (simple poles)
5. $\pm 1, \pm i$ (third-order poles), ∞ (essential singularity)
7. $-i, 2i$ (second-order poles), ∞ (essential singularity)
9. 0 (simple pole), ∞ (essential singularity)
11. 0 (essential singularity), $\pm 2/\pi, \pm 2/3\pi, \cdots$ (simple poles)
13. ∞ (essential singularity)
15. $1, \infty$ (essential singularities), $\pm 2n\pi i$ ($n = 0, 1, \cdots$, simple poles)
17. $\pm 2, \pm 2i$ (second order)
19. 0 (third order), $\pm 1, \pm 2, \cdots$ (second order)
21. $(2n+1)/2$ (third order)
23. $\pm 1/\sqrt{3}$ (simple)
25. $0, \pm 2\pi, \pm 4\pi, \cdots$ (fourth order), by (6), Sec. 12.7
27. $f(z) = (z-z_0)^n g(z)$ by (3), and $g(z_0) \neq 0$. Hence $1/g(z_0)$ is analytic at $z = z_0$. Let its Taylor series be $\frac{1}{g(z)} = c_0 + c_1(z-z_0) + \cdots$. Then $\frac{1}{f(z)} = \frac{c_0 + c_1(z-z_0) + \cdots}{(z-z_0)^n}$, which proves the first statement. Multiplication of $h(z)$ does not change this.
29. Apply Theorem 3 to $f(z) - k$.
31. Region between the 0° and 90° meridians
33. Small spherical disk centered at the North Pole
35. Belt between two parallels that includes the equator

CHAPTER 14 (REVIEW QUESTONS AND PROBLEMS), page 834

21. ∞, $\sin(z-2)$
23. 1/2, $\operatorname{Ln}(1+2z)$
25. 1/3
27. ∞, e^{-z^2}
29. ∞, $\cosh\sqrt{z}$
31. $1 - 2z + (2z)^2/2! - (2z)^3/3! + - \cdots, \quad R = \infty$
33. $1/2 + (z+1)/4 + (z+1)^2/8 + (z+1)^3/16 + \cdots, \quad R = 2$
35. $k - k^2(z-2-3i) + k^3(z-2-3i)^2 - + \cdots, k = (2-3i)/13, R = \sqrt{13}$
37. $1 + 3z + 6z^2 + 10z^3 + \cdots, \quad R = 1$
39. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\left(z-\frac{\pi}{2}\right)^{2n-2}, \quad \left|z-\frac{\pi}{2}\right| > 0$, pole of second order
41. $\sum_{n=0}^{\infty} z^{n-4}, \quad 0 < |z| < 1$, pole of fourth order
43. $\sum_{n=0}^{\infty}\frac{1}{(2n)!z^{2n-3}}, \quad |z| > 0$, essential singularity
45. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^{n-3}, \quad 0 < |z-1| < 1$, pole of second order
47. $\sum_{n=1}^{\infty}\frac{1}{n!n} z^{n-2}, \quad |z| > 0$, simple pole
49. $\sum_{n=0}^{\infty}\frac{e^i}{n!}(z-i)^{n-5}, \quad |z-i| > 0$, pole of fifth order

PROBLEM SET 15.1, page 841

1. -3 (at $z = 1$)
3. $-1/3!$ (at $z = 0$)
5. $i/3$ (at $-i$), $-16i/3$ (at $2i$)
7. 1 (at $z = \pm n\pi$)
9. $\mp\frac{1}{2}$ (at ± 1) by (5)
11. 1 (at $z = 0$)
13. $-9/16, 9/16, -9i/16, 9i/16$ (poles of second order at $z = 1, -1, i, -i$)
15. 3/17 (at $z = -1$)
17. 0
19. $2\pi i$
21. $-6\pi i$
23. $(1 - \pi^3/128)\pi i$

PROBLEM SET 15.2, page 844

1. $-2\pi i$ 3. $-6\pi i$ 5. 0 7. $10\pi i$ 9. $4\pi i/5$
11. $2\pi i(\cosh 1 - 1) = 3.412i$ 13. $2\pi i$ 15. $\pi/3$
17. πi 19. $8\pi i$ 21. $\pi i \sin \frac{1}{2} = 1.506i$ 23. $-4i$
25. $-4i$ 27. 0 29. $-4i \sinh \frac{1}{2} = -2.084i$

PROBLEM SET 15.3, page 849

1. $4\pi/\sqrt{3}$ 3. $2\pi/35$ 5. $8\pi/3$ 7. $\pi/30$ 9. 0
11. π 13. $\pi/\sqrt{2}$ 15. $2\pi/3$ 17. $\pi/2$ 19. $\pi/16$

PROBLEM SET 15.4, page 855

3. 0 5. $(2\pi/\sqrt{3})e^{-\sqrt{3}/2} \cos \frac{1}{2} = 1.339$ 7. π/e
9. 0 11. $-\pi(\sin 1)/\sqrt{3}e^{\sqrt{3}} = -0.2700$
13. $\pi/10e^2 - 2\pi/30e^3$ 15. π 17. $-\sqrt{3}\pi/6$
19. $-\pi/2$ 21. $\pi/2$ 23. $(3 - e^{-\pi})\pi/2 = 4.645$

CHAPTER 15 (REVIEW QUESTIONS AND PROBLEMS), page 856

11. 0, yes 13. $5\pi/2$, yes 15. $-18\pi i$, yes 17. 0, no
19. $-2\pi i/(n+1)!$, yes 21. $-i/2$, no 23. $-64\pi i$, yes
25. $\pi i/4$, yes 27. $2\pi/7$ 29. $4\pi/\sqrt{3}$ 31. $\pi/60$
33. $\pi/2$ 35. 0 37. $\pi/3$ 39. $\pi/2$

PROBLEM SET 16.1, page 863

1. $u = 1 - \frac{1}{4}v^2, 4 - \frac{1}{16}v^2, 9 - \frac{1}{36}v^2, 16 - \frac{1}{64}v^2$ 3. $v = 20$
5. The positive and the negative v-axis, respectively 7. $|w| < \frac{1}{9}$
9. $|\arg w| < 2\pi/3$ 11. $\pi/2 < \arg w < \pi$ 13. $4e^{it}, 4ie^{it}$
15. $3 \cos t + i \sin t, \quad -3 \sin t + i \cos t$
17. $t + it^{-1}, 1 - it^2$ 19. $-a/2$ 21. $\pm 2, \pm 2i$
23. $0, \pm \pi i/2, \pm \pi i, \cdots$ 25. By conformality 27. Only in size

PROBLEM SET 16.2, page 868

1. $z = 0$ 3. $0, \pm 1, \pm i$ 5. $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ 7. $\pm i$
9. $\pm i$ 11. $w = 4/z$, $w = (z + 4)/(z + 1)$, etc. 13. $w = az/d$
15. $w = (az + b)/(a - bz)$ 17. $z = (2iw - 4i)/(-w + 3)$

PROBLEM SET 16.3, page 873

1. $w = 3z + 2$ 3. $w = (z + 1)/(-3z + 1)$
5. $w = 1/z$ 7. $w = (3iz + 1)/z$
9. $w = iz$ 13. $z = (-4w + 1)/(2w - 1)$
15. $w = (2z - i)/(-iz - 2)$ 19. $w = (z^4 - i)/(-iz^4 + 1)$

PROBLEM SET 16.4, page 878

1. $1 \leqq |w| \leqq e, 0 \leqq \arg w \leqq \frac{1}{2}\pi$ 3. $e < |w| < e^2, -\frac{1}{2}\pi < \arg w < \frac{1}{2}\pi$
5. Interior of the ellipse $u^2/(\cosh^2 2) + v^2/(\sinh^2 2) = 1$ in the first quadrant
7. Elliptical annulus bounded by $u^2/\cosh^2 1 + v^2/\sinh^2 1 = 1$ and $u^2/\cosh^2 2 + v^2/\sinh^2 2 = 1$ and cut along the positive imaginary axis
9. $t = z^2$ maps the given region onto the strip $0 < \text{Im } t < \pi$, and $w = e^t$ maps this strip onto the upper half-plane. *Ans.* $w = e^{z^2}$
11. $w' = \cos z = 0$ at $z = \pm(2n + 1)\pi/2, n = 0, 1, \cdots$
13. $\pm n\pi i, n = 0, 1, \cdots$ 15. Upper half-plane $v > 0$
17. Lower half-plane $v < 0$ 19. $\ln 2 \leqq u \leqq \ln 3, \pi/4 \leqq v \leqq \pi/2$

PROBLEM SET 16.5, page 882

1. w moves once around the unit circle $|w| = 1$.
5. $|z| = 1$; $\ln z = \ln |z| + i\theta = i\theta$ moves up the v-axis by 2π each time.
7. ± 1 (first order), 2 sheets 9. 0, 2 sheets
11. a, 3 sheets 13. $0, \pm 1$, 2 sheets
15. -1, infinitely many sheets 17. $\frac{1}{2}i$, 3 sheets
19. 0, 2 sheets

CHAPTER 16 (REVIEW QUESTIONS AND PROBLEMS), page 883

11. $u = \frac{1}{4}v^2 - 1, \frac{1}{4}v^2 - 1$ 13. $|w| = 6.25, |\arg w| < \pi/4$
15. $v = 4u/3$ 17. $\pi/4 < \arg w \leqq \pi/2$
19. The domain between the parabolas $u = \frac{1}{4} - v^2$ and $u = 1 - \frac{1}{4}v^2$
21. $|\arg w| < \pi/8$ 23. $u = 1$ 25. $|w + \frac{1}{2}| = \frac{1}{2}$ 27. $0, \pm i$
29. $\pm i/\sqrt{3}$ 31. $(\pm n + \frac{1}{4})\pi$ 33. iz 35. $z/(z + 2)$
37. $5z$ 39. $2z/(z - 1)$ 41. $w = iz^3 + 1$ 43. $w = iz$
45. $w = e^{3z}$ 47. $w = z^2/2k$ 49. $2 \pm \sqrt{6}$

PROBLEM SET 17.1, page 889

1. $\Phi = 20x + 200$ 3. $\Phi = 20(1 - y/d)$
5. $\Phi = 110 - 50xy$ 7. $(110 \ln r)/\ln 2$
9. $200 - (100 \ln r)/\ln 2$ 11. $y = x/2 + c$
13. $x^2 - y^2 = const$ 15. $(x - 1/2c)^2 + y^2 = 1/4c^2$
17. $u = c \text{ Re } [\text{Ln } (z - a) + \text{Ln } (z + a)] = c \ln |z^2 - a^2|$

PROBLEM SET 17.2, page 893

3. $\Phi(x, y) = U_2xy$. $w = u + iv = iz^2$ maps R onto $-2 \leqq u \leqq 0$.
5. Apply $w = z^2$.
7. Corresponding rays in the w-plane make equal angles, and the mapping is conformal.
9. $z = (2Z - i)/(-iZ - 2)$