usion of this introductory section we should say something about tence of the Laplace transform. Roughly and intuitively speaking ation is as follows. For a fixed s the integral in (1) will exist if the ntegrand $e^{-st}f(t)$ goes to zero fast enough as $t \to \infty$, say, at least like nential function with a negative exponent. This motivates the in (2) in the subsequent existence theorem. $f(t)$ need not be continuous of practical importance since discontinuous inputs (driving forces) those for which the Laplace transform method becomes particularly t suffices to require that $f(t)$ be piecewise continuous on every finite in the range $t \geqq 0$.

finition, a function $f(t)$ is **piecewise continuous** on a finite interval b if $f(t)$ is defined on that interval and is such that the interval can vided into finitely many intervals, in each of which $f(t)$ is continuous finite limits as t approaches either endpoint of the interval of sub from the interior.

ows from this definition that finite jumps are the only discontinuities ecewise continuous function may have; these are known as *ordinary uities*. Figure 87 shows an example. Clearly, the class of piecewise us functions includes every continuous function.

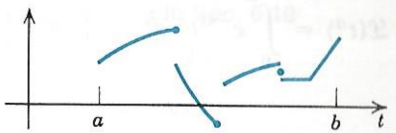


Fig. 87. Example of a piecewise continuous function $f(t)$ (The dots mark the function values at the jumps.)

ce theorem for Laplace transforms)

be a function that is piecewise continuous on every finite interval nge $t \geqq 0$ and satisfies

$$|f(t)| \leqq Me^{\gamma t} \qquad \textit{for all } t \geqq 0$$

ome constants γ and M. Then the Laplace transform of $f(t)$ exists $> \gamma$.

ince $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any rval on the t-axis. From (2), assuming that $s > \gamma$, we obtain

$$= \left| \int_0^\infty e^{-st} f(t)\, dt \right| \leqq \int_0^\infty |f(t)| e^{-st}\, dt \leqq \int_0^\infty Me^{\gamma t} e^{-st}\, dt = \frac{M}{s-\gamma}$$

e condition $s > \gamma$ was needed for the existence of the last integral. pletes the proof. ■

form (2). For example,

$$(3) \qquad \cosh t < e^t, \qquad t^n < n!\, e^t \quad (n = 0, 1, \cdots) \qquad \text{for all } t > 0,$$

and any function that is bounded in absolute value for all $t \geqq 0$, such as the sine and cosine functions of a real variable, satisfies that condition. An example of a function that does not satisfy a relation of the form (2) is the exponential function e^{t^2}, because, no matter how large we choose M and γ in (2),

$$e^{t^2} > Me^{\gamma t} \qquad \text{for all } t > t_0$$

where t_0 is a sufficiently large number, depending on M and γ.

It should be noted that the conditions in Theorem 2 are sufficient rather than necessary. For example, the function $1/\sqrt{t}$ is infinite at $t = 0$, but its transform exists; in fact, from the definition and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [see (30) in Appendix 3] we obtain

$$\mathcal{L}(t^{-1/2}) = \int_0^\infty e^{-st} t^{-1/2}\, dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2}\, dx = \frac{1}{\sqrt{s}} \Gamma(\tfrac{1}{2}) = \sqrt{\frac{\pi}{s}}.$$

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at various isolated points (see Ref. [A10] in Appendix 1). Since this is of no importance in applications, we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical. Of course, this *is* of practical importance. Why? (Remember the introduction to the chapter.)

Problem Set 6.1

Find the Laplace transforms of the following functions. (a, b, T, ω, and θ are constants.)

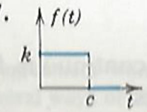
1. $3t + 4$ **2.** $at + b$ **3.** $t^2 + at + b$ **4.** $(a + bt)^2$

5. $\cos(\omega t + \theta)$ **6.** $\sin(\omega t + \theta)$ **7.** $\sin(2n\pi t/T)$ **8.** $\sin^2 t$

9. $\cos^2 \omega t$ **10.** $\sin^2 \omega t$ **11.** $\cos^2 t$ **12.** $-5\cos 0.4t$

13. e^{at+b} **14.** $\cosh^2 3t$ **15.** $\sinh^2 2t$ **16.** $\sin t \cos t$

17.



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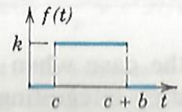

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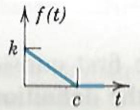
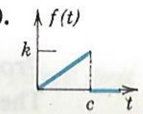


20.



21. Obtain formula 6 in Table 6.1 from formulas 9 and 10.
22. Obtain formula 10 in Table 6.1 from formula 6.
23. Derive formulas 7 and 8 in Table 6.1 by integration by parts.
24. Obtain the answer to Prob. 20 from the answers to Probs. 17 and 19.
25. Using $\cos x = \cosh ix$ and $\sin x = -i \sinh ix$, derive formulas 7 and 8 in Table 6.1 from formulas 9 and 10.
26. Using Prob. 17, find $\mathcal{L}(f)$, where $f(t) = 0$ if $t \leqq 4$, $f(t) = 1$ if $t > 4$.

Find $f(t)$ if $F(s) = \mathcal{L}(f)$ is as follows. (a, b, etc. are constants.)

27. $\dfrac{5}{s+3}$ 28. $\dfrac{s-9}{s^2-9}$ 29. $\dfrac{1}{s^2+25}$ 30. $\dfrac{4}{(s+1)(s+2)}$

31. $\dfrac{1}{s^4}$ 32. $\dfrac{s+1}{s^2+1}$ 33. $\dfrac{1}{s(s+1)}$ 34. $\dfrac{n\pi L}{L^2s^2+n^2\pi^2}$

35. $\dfrac{9}{s^2+3s}$ 36. $\dfrac{4(s+1)}{s^2-16}$ 37. $\dfrac{2}{s}+\dfrac{1}{s+2}$ 38. $\dfrac{0.25}{s-5}-\dfrac{0.4}{s^3}$

39. Prove (3).
40. **(Linearity of the inverse Laplace transform)** Show that $\mathcal{L}^{-1}$ is linear. *Hint.* Use that $\mathcal{L}$ is linear.

Transforms of Derivatives and Integrals

In this section we discuss and apply the most crucial property of the Laplace transform, namely, that, roughly speaking, differentiation of functions corresponds to the multiplication of transforms by s, and integration of functions corresponds to the division of transforms by s. Hence ***the Laplace transform replaces operations of calculus by operations of algebra on transforms.*** This, in a nutshell, is Laplace's basic idea, for which we should admire him.

Our program for this section is as follows. Theorem 1 concerns the differentiation of $f(t)$, Theorem 2 the extension to higher derivatives, and Theorem 3 the integration of $f(t)$. We also include examples as well as a first application to a differential equation.

[Laplace transform of the derivative of $f(t)$]
Suppose that $f(t)$ is continuous for all $t \geqq 0$, satisfies (2), Sec. 6.1, for some γ and M, and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geqq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > \gamma$, and

$$\mathcal{L}(f') = \int_0^\infty e^{-st}f'(t)\,dt = [e^{-st}f(t)]\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt.$$

Since f satisfies (2), Sec. 6.1, the integrated portion on the right is zero at the upper limit when $s > \gamma$, and at the lower limit it is $-f(0)$. The last integral is $\mathcal{L}(f)$, the existence for $s > \gamma$ being a consequence of Theorem 2 in Sec. 6.1. This proves that the expression on the right exists when $s > \gamma$, and is equal to $-f(0) + s\mathcal{L}(f)$. Consequently, $\mathcal{L}(f')$ exists when $s > \gamma$, and (1) holds.

If the derivative $f'(t)$ is merely piecewise continuous, the proof is quite similar; in this case, the range of integration in the original integral must be broken up into parts such that f' is continuous in each such part. ∎

REMARK This theorem may be extended to piecewise continuous functions $f(t)$, but in place of (1) we then obtain the formula (1*) in Prob. 40 at the end of the current section.

By applying (1) to the second derivative $f''(t)$ we obtain

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0)$$
$$= s[s\mathcal{L}(f) - f(0)] - f'(0);$$

that is,

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Similarly,

$$\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2f(0) - sf'(0) - f''(0), \tag{3}$$

etc. By induction we thus obtain the following extension of Theorem 1.

Theorem 2 **(Laplace transform of the derivative of any order n)**
Let $f(t)$ and its derivatives $f'(t), f''(t), \cdots, f^{(n-1)}(t)$ be continuous functions for all $t \geqq 0$, satisfying (2), Sec. 6.1, for some γ and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geqq 0$. Then the Laplace transform of $f^{(n)}(t)$ exists when $s > \gamma$, and is given by

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{4}$$

EXAMPLE 1 Let $f(t) = t^2$. Find $\mathcal{L}(f)$.

$$e^{-as}F(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)\,dt = \mathcal{L}\{f(t-a)u(t-a)\}.$$

is fair to say that we are already approaching the stage where we can ck problems for which the Laplace transform method is preferable to usual method, as the examples in the next section will illustrate. In this nection we need the transform of the unit step function $u(t-a)$,

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s > 0).$$

s formula follows directly from the definition because

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt$$

$$= \int_0^a e^{-st}0\,dt + \int_a^\infty e^{-st}1\,dt = -\frac{1}{s}e^{-st}\Big|_a^\infty .$$

et us consider two simple examples. Further applications follow in the blem set and in the next sections.

lication of Theorem 2

the inverse transform of e^{-3s}/s^3.

ion. Since $\mathcal{L}^{-1}(1/s^3) = t^2/2$ (see Table 6.1 in Sec. 6.1), Theorem 2 gives (Fig. 93)

$$\mathcal{L}^{-1}(e^{-3s}/s^3) = \tfrac{1}{2}(t-3)^2u(t-3) = \begin{cases} 0 & \text{if } t < 3 \\ \tfrac{1}{2}(t-3)^2 & \text{if } t > 3. \end{cases}$$

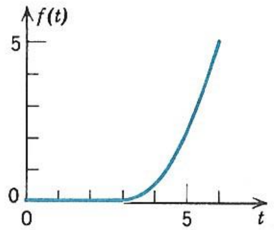


Fig. 93. Example 4

of unit step functions

he transform of the function (Fig. 94)

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ \sin t & \text{if } \quad t > 2\pi. \end{cases}$$

in; so we add $u(t-2\pi)\sin t$. Together,

$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin t.$$

2nd Step. The last term equals $u(t-2\pi)\sin(t-2\pi)$ because of the periodicity, so that (5), (4), and Table 6.1 (Sec. 6.1) give

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}.$$

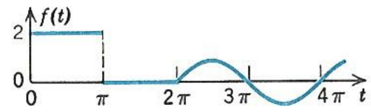


Fig. 94. Example 5

EXAMPLE 6 Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{se^{-\pi s}}{s^2+1}.$$

Solution. From Table 6.1 (Sec. 6.1) and the Theorem 2,

$$f(t) = 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)\,u(t-\pi)$$

$$= 2t - 2tu(t-2) - \cos t\,u(t-\pi) = \begin{cases} 2t & \text{if } 0 < t < 2 \\ 0 & \text{if } 2 < t < \pi \\ -\cos t & \text{if } \quad t > \pi. \end{cases}$$

Problem Set 6.3

Applications of the First Shifting Theorem

Find the Laplace transforms of the following functions.

1. $4.5te^{3.5t}$
2. t^2e^{-2t}
3. $e^t \sin t$
4. $e^{-t}\cos t$
5. $e^{-t}\sin(\omega t + \theta)$
6. $e^{-\alpha t}\sin \omega t$
7. $2t^3e^{-t/2}$
8. $e^{-2t}(2\cos 3t - \sin 3t)$
9. $e^{-\alpha t}(A\cos \beta t + B\sin \beta t)$
10. $e^{-t}(a_0 + a_1t + \cdots + a_nt^n)$

Find $f(t)$ if $\mathcal{L}(f)$ equals

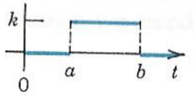
11. $\dfrac{\pi}{(s+\pi)^2}$
12. $\dfrac{1}{(s+\frac{1}{2})^3}$
13. $\dfrac{s-2}{s^2-4s+5}$
14. $\dfrac{1}{s^2+2s+5}$
15. $\dfrac{s}{(s+3)^2+1}$
16. $\dfrac{as+b}{(s+c)^2+\omega^2}$
17. $\dfrac{6}{s^2-4s-5}$
18. $\dfrac{2}{s^2+s+\frac{1}{2}}$

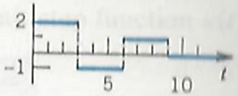
Representing the hyperbolic functions in terms of exponential functions and applying the first shifting theorem, show that

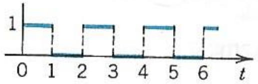
19. $\mathcal{L}(\cosh at \cos at) = \dfrac{s^3}{s^4 + 4a^4}$ 20. $\mathcal{L}(\cosh at \sin at) = \dfrac{a(s^2 + 2a^2)}{s^4 + 4a^4}$

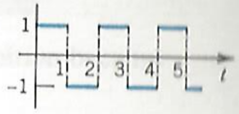
21. $\mathcal{L}(\sinh at \cos at) = \dfrac{a(s^2 - 2a^2)}{s^4 + 4a^4}$ 22. $\mathcal{L}(\sinh at \sin at) = \dfrac{2a^2 s}{s^4 + 4a^4}$

Unit step function. Represent the following functions in terms of unit step functions and find their Laplace transforms.

23. 

24. 

25. 

26. 

Applications of the Second Shifting Theorem

Sketch the following functions and find their Laplace transforms.

27. $(t - 1)u(t - 1)$ 28. $tu(t - 1)$ 29. $(t - 1)^2u(t - 1)$ 30. $t^2u(t - 1)$
31. $e^t u(t - \frac{1}{2})$ 32. $u(t - 1)\cosh t$ 33. $u(t - \pi)\cos t$ 34. $u(t - \frac{1}{2}\pi)\sin t$

In each case sketch the given function, which is assumed to be zero outside the given interval, and find its Laplace transform.

35. $t \quad (0 < t < 1)$
36. $t \quad (0 < t < 2)$
37. $e^t \quad (0 < t < 1)$
38. $t^2 \quad (0 < t < 3)$
39. $t \quad (0 < t < a)$
40. $\sin t \quad (2\pi < t < 4\pi)$
41. $2 \cos \pi t \quad (1 < t < 2)$
42. $1 - e^{-t} \quad (0 < t < \pi)$

Find and sketch the inverse Laplace transforms of the following functions.

43. e^{-3s}/s^2
44. $e^{-\pi s}/(s^2 + 2s + 2)$
45. $3(e^{-4s} - e^{-s})/s$
46. $(e^{-2s} - e^{-4s})/(s - 2)$
47. $se^{-\pi s}/(s^2 + 4)$
48. $e^{-s}/(s^2 + \omega^2)$
49. e^{-s}/s^4
50. $(1 - e^{-\pi s})/(s^2 + 4)$

Initial value problems. Using Laplace transforms, solve:

51. $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$
52. $y'' - 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 2$
53. $4y'' - 4y' + 37y = 0, \quad y(0) = 3, \quad y'(0) = 1.5$
54. $9y'' - 6y' + y = 0, \quad y(0) = 3, \quad y'(0) = 1$

57. $y'' + 6y' + 8y = -e^{-3t} + 3e^{-5t}, \quad y(0) = 4, \quad y'(0) = -14$
58. $y'' + 2y' + 5y = 9\cosh 2t + 4\sinh 2t, \quad y(0) = 1, \quad y'(0) = 2$
59. $y'' + y = r(t), \quad r(t) = t$ if $0 < t < 1$ and 0 if $t > 1$, $\quad y(0) = y'(0) = 0$
60. $y'' - 5y' + 6y = r(t), \quad r(t) = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$, $\quad y(0) = 1$, $y'(0) = -2$
61. $y'' + 9y = r(t), \quad r(t) = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$, $\quad y(0) = 0$, $y'(0) = 4$
62. $y'' + 3y' + 2y = r(t), \quad r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$, $y(0) = y'(0) = 0$
63. $y'' + 4y = r(t), \quad r(t) = 3 \sin t$ if $0 < t < \pi$ and $-3 \sin t$ if $t > \pi$, $y(0) = 0, \quad y'(0) = 3$
64. $y'' + 2y' + 2y = r(t), \quad r(t) = t$ if $0 < t < 1$ and $2t^2$ if $t > 1$, $\quad y(0) = \frac{1}{2}$, $y'(0) = -\frac{1}{2}$
65. $y'' + y' - 2y = r(t), \quad r(t) = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi$, $\quad y(0) = 1, \quad y'(0) = -0$

Models of electric circuits

66. A capacitor of capacitance C is charged so that its potential is V_0. At $t = 0$ the switch in Fig. 95 is closed and the capacitor starts to discharge through the resistor of resistance R. Using Laplace transforms, find the charge $q(t)$ on the capacitor.
67. Find the current $i(t)$ in the circuit in Fig. 96, assuming that no current flows when $t \leqq 0$ and the switch is closed at $t = 0$.

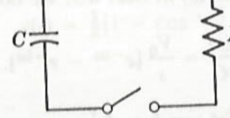


Fig. 95. Problem 66

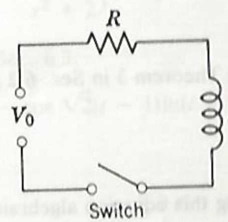
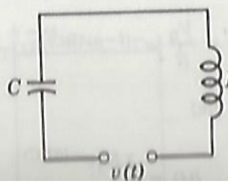


Fig. 96. Problem 67

Find the current $i(t)$ in the LC-circuit shown in Fig. 97, assuming $L = 1$ henry, $C = 1$ farad, zero initial current and charge on the capacitor, and $v(t)$ as follows.

68. $v = 1$ if $0 < t < a$ and 0 otherwise
69. $v = t$ if $0 < t < 1$ and $v = 1$ if $t > 1$
70. $v = 1 - e^{-t}$ if $0 < t < \pi$ and 0 otherwise



but an ordinary function which is everywhere 0 except at a single point must have the integral 0. Nevertheless, in impulse problems it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function.

EXAMPLE 4 Response of a damped vibrating system to a unit impulse

Determine the response of the damped mass–spring system (see Sec. 2.11) governed by

$$y'' + 3y' + 2y = \delta(t-a), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Thus the system is initially at rest and at time $t = a$ is suddenly given a sharp hammerblow.

Solution. By (4) we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = e^{-as}.$$

Solving for Y, we have

$$Y(s) = F(s)e^{-as}, \qquad \text{where} \qquad F(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse transform, we obtain

$$f(t) = \mathcal{L}^{-1}(F) = e^{-t} - e^{-2t}.$$

Hence by the second shifting theorem (Sec. 6.3) we have

$$y(t) = \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } 0 \le t < a \\ e^{-(t-a)} - e^{-2(t-a)} & \text{if } t > a. \end{cases}$$

Figure 103 shows this solution for $a = 1$. The reader may compare this with the output in Example 3 (Fig. 101) and comment. ■

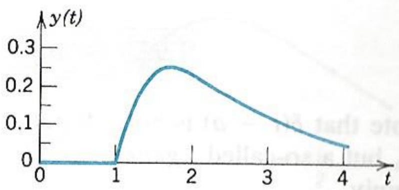


Fig. 103. Output in Example 4 with $a = 1$

Problem Set 6.4

***RC*-circuit.** Find the current in the *RC*-circuit in Fig. 104 with $R = 100$ ohms, $C = 0.1$ farad, and electromotive force $v(t)$ [volts] as follows. Assume that the circuit is quiescent before $v(t)$ is applied.

1. $v(t) = 100$ if $1 < t < 2$ and 0 otherwise.
2. $v(t) = 10000$ if $1 < t < 1.01$ and 0 otherwise
3. $v(t) = e^{-t}$ if $t > 2$ and 0 otherwise
4. $v(t) = 50(t-3)$ if $t > 3$ and 0 otherwise
5. $v(t) = 200t$ if $0 < t < 1$ and 0 if $t > 1$

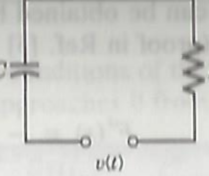


Fig. 104. *RC*-circuit in Probs. 1–5

Effect of the delta function on vibrating systems and in other initial value problems. Solve:

6. $y'' + 4y = \delta(t-\pi), \quad y(0) = 2, \quad y'(0) = 0$
7. $y'' + 9y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$
8. $y'' - y = \sin t + \delta(t - \frac{1}{2}\pi), \quad y(0) = 3.0, \quad y'(0) = -3.5$
9. $y'' + y = \delta(t-\pi) - \delta(t-2\pi), \quad y(0) = 0, \quad y'(0) = 1$
10. $y'' - y = -2\sin t + \delta(t-1), \quad y(0) = 0, \quad y'(0) = 2$
11. $y'' + 2y' + 2y = \delta(t-2\pi), \quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = \delta(t-4), \quad y(0) = y'(0) = 0$
13. $y'' + 4y' + 5y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 3$
14. $y'' + 2y' - 3y = \delta(t-2) + \delta(t-3), \quad y(0) = 1, \quad y'(0) = 0$
15. $y'' + 2y' - 3y = -8e^{-t} - \delta(t - \frac{1}{2}), \quad y(0) = 3, \quad y'(0) = -5$
16. $y'' + 4y' + 4y = 1 + t + \delta(t-1), \quad y(0) = 0, \quad y'(0) = 2.25$
17. $y'' + 2y' + 5y = 8e^t + \delta(t-1), \quad y(0) = 2, \quad y'(0) = 0$
18. $y'' + 5y' + 6y = u(t-1) + \delta(t-2), \quad y(0) = 0, \quad y'(0) = 1$
19. $y'' + 2y' + 5y = 25t - \delta(t-\pi), \quad y(0) = -2, \quad y'(0) = 5$
20. $y'' - y' - 2y = -4t^2 + \delta(t-2), \quad y(0) = 5, \quad y'(0) = -1$

6.5 Differentiation and Integration of Transforms

The number of methods for obtaining transforms or inverse transforms and their application in solving differential equations is surprisingly large. They include direct integration (Sec. 6.1), the use of linearity (Sec. 6.1), shifting (Sec. 6.3), and differentiation or integration of original functions $f(t)$ (Sec. 6.2). But this is not all: in this section we consider differentiation and integration of *transforms* $F(s)$ and find out the corresponding operations for original functions $f(t)$.

Differentiation of Transforms

It can be shown that if $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1, then the derivative of its transform

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2}$$

$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2}.$$

The representations in terms of partial fractions are

$$Y_1 = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}, \qquad Y_2 = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos \sqrt{k}t + \sin \sqrt{3k}t$$

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos \sqrt{k}t - \sin \sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" and a "rapid" oscillation. ■

Problem Set 6.7

Using partial fractions, find $f(t)$ if $\mathcal{L}(f)$ equals

1. $\dfrac{1}{(s-4)(s-1)}$ **2.** $\dfrac{s-3}{s^2-1}$ **3.** $\dfrac{3s}{s^2+2s-8}$

4. $\dfrac{s+12}{s^2+4s}$ **5.** $\dfrac{s+13}{s^2+2s+10}$ **6.** $\dfrac{-s+4.5}{s^2+2.25}$

7. $\dfrac{s^2-6s+4}{s^3-3s^2+2s}$ **8.** $\dfrac{s}{(s-2)^3}$ **9.** $\dfrac{10-4s}{(s-2)^2}$

10. $\dfrac{s}{s^2+2s+2}$ **11.** $\dfrac{s^2+s-2}{(s+1)^3}$ **12.** $\dfrac{s^2+2s}{(s^2+2s+2)^2}$

13. $\dfrac{-2s^3+26s}{s^4-10s^2+9}$ **14.** $\dfrac{s+1}{s^2+4s+13}$ **15.** $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$

16. $\dfrac{6s^2-26s+26}{s^3-6s^2+11s-6}$ **17.** $\dfrac{s^3+6s^2+14s}{(s+2)^4}$ **18.** $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3}$

19. $\dfrac{2s^2-3s}{(s-2)(s-1)^2}$ **20.** $\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}$

21. Solve Prob. 1 by convolution.

22. Check the result in Example 1 by (i) working backward, (ii) using Theorem 3, Sec. 6.2, (iii) convolution methods.

Some inverses in terms of hyperbolic functions. Show that

23. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4+4a^4}\right\} = \dfrac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$

24. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\} = \dfrac{1}{2a^2}\sinh at \sin at$

25. $\ldots \, s^4 + 4a^4 \, \ldots \, 2a \, (\cosh at \sin at + \sinh at \cos at)$

26. $\mathcal{L}^{-1}\left\{\dfrac{s^3}{s^4+4a^4}\right\} = \cosh at \cos at$

Systems of differential equations. Solve the following initial value problem by means of Laplace transforms.

27. $y_1' = -y_2, \quad y_2' = y_1, \quad y_1(0) = 1, \quad y_2(0) = 0$

28. $y_1' + y_2 = 2\cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$

29. $y_1' = -y_1 + y_2, \quad y_2' = -y_1 - y_2, \quad y_1(0) = 1, \quad y_2(0) = 0$

30. $y_1' = 6y_1 + 9y_2, \quad y_2' = y_1 + 6y_2, \quad y_1(0) = -3, \quad y_2(0) = -3$

31. $y_1' = 2y_1 + 4y_2, \quad y_2' = y_1 + 2y_2, \quad y_1(0) = -4, \quad y_2(0) = -4$

32. $y_1' = -y_1 + 4y_2, \quad y_2' = 3y_1 - 2y_2, \quad y_1(0) = 3, \quad y_2(0) = 4$

33. $y_1' = 2y_1 - 4y_2, \quad y_2' = y_1 - 3y_2, \quad y_1(0) = 3, \quad y_2(0) = 0$

34. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2, \quad y_1(0) = -3, \quad y_2(0) = 7$

35. $y_1' = -2y_1 + 3y_2, \quad y_2' = 4y_1 - y_2, \quad y_1(0) = 4, \quad y_2(0) = 3$

36. $y_1'' + y_2 = -5\cos 2t, \quad y_2'' + y_1 = 5\cos 2t,$
$y_1(0) = 1, \quad y_1'(0) = 1, \quad y_2(0) = -1, \quad y_2'(0) = 1$

37. $y_1'' = y_1 + 3y_2, \quad y_2'' = 4y_1 - 4e^t,$
$y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$

38. $y_1'' = -5y_1 + 2y_2, \quad y_2'' = 2y_1 - 2y_2,$
$y_1(0) = 3, \quad y_2(0) = 1, \quad y_1'(0) = y_2'(0) = 0$

39. $y_1' + y_2' = 2\sinh t, \quad y_2' + y_3' = e^t, \quad y_3' + y_1' = 2e^t + e^{-t},$
$y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0$

40. $2y_1' - y_2' - y_3' = 0, \quad y_1' + y_2' = 4t + 2, \quad y_2' + y_3 = t^2 + 2,$
$y_1(0) = y_2(0) = y_3(0) = 0$

6.8 Periodic Functions. Further Applications

Periodic functions appear in many practical problems, and in most cases they are more complicated than just single cosine or sine functions. This justifies the topic of the present section, which is a systematic approach to the transformation of periodic functions. The text and the problem set also include further applications.

Let $f(t)$ be a function that is defined for all positive t and has the period p (> 0), that is,

$$f(t + p) = f(t) \qquad \text{for all } t > 0.$$

wtooth wave
nd the Laplace transform of the function (Fig. 111)

$$f(t) = \frac{k}{p}t \quad \text{if } 0 < t < p, \qquad f(t+p) = f(t).$$

lution. Integration by parts yields

$$\int_0^p e^{-st} t\, dt = -\frac{t}{s}e^{-st}\Big|_0^p + \frac{1}{s}\int_0^p e^{-st}\, dt$$
$$= -\frac{p}{s}e^{-sp} - \frac{1}{s^2}(e^{-sp} - 1),$$

d from (1) we thus obtain the result

$$\mathcal{L}(f) = \frac{k}{ps^2} - \frac{ke^{-ps}}{s(1 - e^{-ps})} \qquad (s > 0).$$

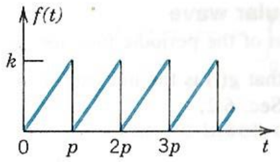


Fig. 111. Sawtooth wave

taircase function
nd the Laplace transform of the staircase function (Fig. 112)

$$g(t) = kn \qquad [np < t < (n+1)p, \quad n = 0, 1, 2, \cdots].$$

lution. Since $g(t)$ is the difference of the functions $h(t) = kt/p$ (whose transform is k/ps^2) and
t) in Example 4, we obtain

$$\mathcal{L}(g) = \mathcal{L}(h) - \mathcal{L}(f) = \frac{ke^{-ps}}{s(1 - e^{-ps})} \qquad (s > 0).$$

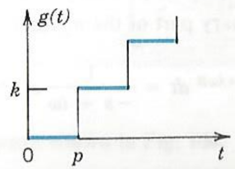


Fig. 112. Staircase function

This is the end of Chap. 6 (except for the tables in Secs. 6.9 and 6.10).
pplication of the Laplace transform to **partial differential equations** is ex-
lained in Sec. 11.13.

This is also the end of Part A on ordinary differential equations. Partial
ifferential equations follow, along with Fourier series, in Chaps. 10 and 11

Problem Set 6.8

Laplace transforms of periodic functions. Sketch the following functions, which are assumed to have the period 2π, and find their transforms.

1. $f(t) = \pi - t \quad (0 < t < 2\pi)$
2. $f(t) = t \quad (0 < t < 2\pi)$
3. $f(t) = 4\pi^2 - t^2 \quad (0 < t < 2\pi)$
4. $f(t) = t^2 \quad (0 < t < 2\pi)$
5. $f(t) = e^t \quad (0 < t < 2\pi)$
6. $f(t) = \sin \frac{1}{2}t \quad (0 < t < 2\pi)$
7. $f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases}$
8. $f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi \\ -1 & \text{if } \pi < t < 2\pi \end{cases}$
9. $f(t) = \begin{cases} t & \text{if } 0 < t < \pi \\ \pi - t & \text{if } \pi < t < 2\pi \end{cases}$
10. $f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ t - \pi & \text{if } \pi < t < 2\pi \end{cases}$
11. How can the answer to Prob. 9 be obtained from the answers to Probs. 7 and 10?
12. Solve Prob. 10 by applying the second shifting theorem (Sec. 6.3) to Prob. 7.
13. Apply Theorem 1 to the function $f(t) = 1$, which is periodic with any period p.

Half-wave and full-wave rectifiers

14. Find the Laplace transform of the half-wave rectification of $-\sin \omega t$ (Fig. 113).

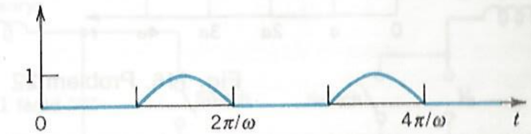


Fig. 113. Problem 14

15. Find the Laplace transform of the full-wave rectification of $\sin \omega t$ (Fig. 114).

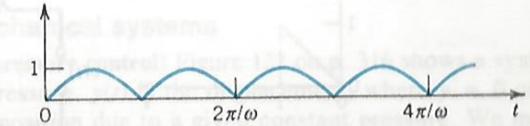


Fig. 114. Problem 15

16. Find the Laplace transform of the full-wave rectification $|\cos \omega t|$ of $\cos \omega t$.
17. Solve Prob. 14 by applying Theorem 2, Sec. 6.3, to the result of Example 3.
18. Check the answer to Prob. 15 by using the results of Example 3 and Prob. 14.

Models of electric circuits

19. Using Laplace transforms, show that the current $i(t)$ in the RLC-circuit in Fig. 115 (constant electromotive force V_0, zero initial current and charge) is

$$i(t) = \begin{cases} (K/\omega^*)e^{-\alpha t} \sin \omega^* t & \text{if } \omega^{*2} > 0 \\ Kte^{-\alpha t} & \text{if } \omega^{*2} = 0 \\ (K/\beta)e^{-\alpha t} \sinh \beta t & \text{if } \omega^{*2} = -\beta^2 < 0; \end{cases}$$

From Table 6.1 in Sec. 6.1 we have

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin \omega t.$$

From this and Theorem 3 we obtain the answer

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right\} = \frac{1}{\omega}\int_0^t \sin \omega\tau \, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

This proves formula 19 in the table in Sec. 6.10. ■

EXAMPLE 8 **Another application of Theorem 3**

Let $\mathcal{L}(f) = \dfrac{1}{s^2(s^2+\omega^2)}$. Find $f(t)$.

Solution. Applying Theorem 3 to the answer in Example 7, we obtain the desired formula

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{1}{s^2+\omega^2}\right)\right\} = \frac{1}{\omega^2}\int_0^t (1 - \cos \omega\tau)\, d\tau = \frac{1}{\omega^2}\left(t - \frac{\sin \omega t}{\omega}\right).$$

This proves formula 20 in the table in Sec. 6.10. ■

Problem Set 6.2

Using (1) or (2), find the transform $\mathcal{L}(f)$ of the given function $f(t)$.

1. $\cos^2 t$ **2.** $\sin^2 \omega t$ **3.** te^{at} **4.** $t \cos t$
5. $\cosh^2 2t$ **6.** $\sinh^2 2t$ **7.** $\cos^2 3t$ **8.** $\cos^2 \pi t$

Further transforms. Using Theorems 1 and 2, derive the following transforms that occur in applications (in connection with resonance, etc.).

9. $\mathcal{L}(t \cos \omega t) = \dfrac{s^2 - \omega^2}{(s^2+\omega^2)^2}$ **10.** $\mathcal{L}(t \sin \omega t) = \dfrac{2\omega s}{(s^2+\omega^2)^2}$

11. $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2-a^2)^2}$ **12.** $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2-a^2)^2}$

Using the formulas in Probs. 9 and 10, show that

13. $\mathcal{L}^{-1}\left(\dfrac{1}{(s^2+\omega^2)^2}\right) = \dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$

14. $\mathcal{L}^{-1}\left(\dfrac{s^2}{(s^2+\omega^2)^2}\right) = \dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$

Application of Theorem 3. Find $f(t)$ if $\mathcal{L}(f)$ equals

15. $\dfrac{3}{s^2+s}$ **16.** $\dfrac{4}{s^3-4s}$ **17.** $\dfrac{4}{s^3+4s}$ **18.** $\dfrac{1}{s^2+as}$

19. $\dfrac{8}{s^4-4s^2}$ **20.** $\dfrac{1}{s}\left(\dfrac{s-a}{s+a}\right)$ **21.** $\dfrac{1}{s^4-2s^3}$ **22.** $\dfrac{1}{s^2}\left(\dfrac{s-a}{s+a}\right)$

23. $\dfrac{2s-\pi}{s^3(s-\pi)}$ **24.** $\dfrac{1}{s^2}\left(\dfrac{s-2}{s^2+4}\right)$ **25.** $\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$ **26.** $\dfrac{1}{s^4(s^2+\pi^2)}$

Initial value problems. Using Laplace transforms, solve:

27. $y'' + 4y = 0,\quad y(0) = 2,\quad y'(0) = -8$

28. $4y'' + \pi^2 y = 0,\quad y(0) = 0,\quad y'(0) = 1$

29. $y'' + \omega^2 y = 0,\quad y(0) = A,\quad y'(0) = B,\quad$ (ω real, not zero)

30. $y'' + 2y' - 8y = 0,\quad y(0) = 0,\quad y'(0) = 6$

31. $y'' + 5y' + 6y = 0,\quad y(0) = 0,\quad y'(0) = 1$

32. $y'' - 4y' + 3y = 2t - \frac{8}{3},\quad y(0) = 0,\quad y'(0) = -\frac{16}{3}$

33. $y'' + 25y = t,\quad y(0) = 1,\quad y'(0) = 0.04$

34. $y'' + 4y = 1 - 2t,\quad y(0) = 0,\quad y'(0) = 0$

35. $y'' + ky' - 2k^2 y = 0,\quad y(0) = 2,\quad y'(0) = 2k$

36. $y'' + \pi^2 y = t^3,\quad y(0) = 6/\pi^4,\quad y'(0) = 0$

Derivation by different methods is possible for various formulas, and is typical of Laplace transforms. Problems 37–39 illustrate this point.

37. Using (1), derive $\mathcal{L}(\sin \omega t)$ from $\mathcal{L}(\cos \omega t)$.

38. Find $\mathcal{L}(\cos^2 t)$ (a) by the use of the result of Example 3, (b) by the method used in Example 3, (c) by expressing $\cos^2 t$ in terms of $\cos 2t$.

39. Check the answer to Prob. 5 by writing $\cosh 2t$ in terms of exponential functions.

40. **(Extension of Theorem 1)** Show that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at $t = a$ (> 0), the other conditions remaining the same as in Theorem 1, then (see Fig. 88)

$$\text{(1*)} \qquad \mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a+0) - f(a-0)]e^{-as}.$$

Using (1*), find the Laplace transform of $f(t) = t$ if $0 < t < 1$, $f(t) = 1$ if $1 < t < 2$, $f(t) = 0$ otherwise.

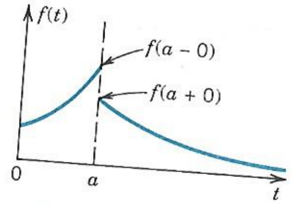


Fig. 88. Formula (1*)

6.3 s-Shifting, t-Shifting, Unit Step Function

What state have we reached and what is our next goal? We know that the Laplace transform is linear (Theorem 1, Sec. 6.1), that differentiation of $f(t)$ roughly corresponds to the multiplication of $\mathcal{L}(f)$ by s (Theorems 1 and 2, Sec. 6.2), and that this property is essential in solving differential equations. A first illustration of the technique is given in Example 5, Sec. 6.2. But there the solution may easily be found by the usual methods.

To provide applications such that the Laplace transform can show its real power, we first have to derive some further general properties of it. Two very important properties concern the shifting on the s-axis and the shifting on the t-axis, as expressed in the two shifting theorems

Our result is

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = \frac{2}{t}(1-\cos \omega t).$$

This proves formula 42 in the table in Sec. 6.10. ∎

EXAMPLE 3 **Integration of transforms**

Reasoning as in Example 2, we obtain (see formula 43 in the table in Sec. 6.10)

$$\mathcal{L}^{-1}\left\{\ln\left(1-\frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1-\cosh at).$$ ∎

Differential Equations with Variable Coefficients

From (1) and (1), (2) in Sec. 6.2 and by differentiating we have

$$(7)\qquad \mathcal{L}(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - sY',$$

$$(8)\qquad \mathcal{L}(ty'') = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2Y' + y(0).$$

Hence if a differential equation has coefficients such as $at + b$, we get a first-order differential equation for Y, which is sometimes simpler than the given equation. But if the latter has coefficients $at^2 + bt + c$, we get, by two applications of (1), a second-order differential equation for Y, and this shows that the Laplace transform method works well only for very special equations with variable coefficients. We illustrate it for an important equation in the following example.

EXAMPLE 4 **Laguerre's differential equation, Laguerre polynomials**

Laguerre's differential equation is (see also Problem Set 5.9)

$$(9)\qquad ty'' + (1-t)y' + ny = 0.$$

We determine a solution of (9) with $n = 0, 1, 2, \cdots$. From (7)–(9) we get

$$-2sY - s^2Y' + y(0) + sY - y(0) - (-Y - sY') + nY = 0.$$

Simplification gives

$$(s - s^2)Y' + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, and integrating (with the constant of integration taken zero), we get

$$(10^*)\qquad \frac{dY}{Y} = \left(\frac{n}{s-1} - \frac{n+1}{s}\right) ds \quad \text{and} \quad Y = \frac{(s-1)^n}{s^{n+1}}.$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and show that

$$(10)\qquad l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots.$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (but we

6.1 and the first shifting theorem,

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}.$$

By (4) in Sec. 6.2,

$$\mathcal{L}\{(t^n e^{-t})^{(n)}\} = \frac{n!s^n}{(s+1)^{n+1}}.$$

Now make another shift and divide by $n!$ to get [see (10) and (10*)]

$$\mathcal{L}(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y.$$ ∎

Problem Set 6.5

Using (1), find the Laplace transform of

1. $2t\cos 2t$ **2.** $4te^{2t}$ **3.** t^2e^t **4.** $t\cosh t$
5. $t\sinh 3t$ **6.** $t^2\sinh 2t$ **7.** $t\sin \omega t$ **8.** $t^2\sin \omega t$
9. $t^2\cos t$ **10.** $t^2\cos \omega t$ **11.** $\frac{1}{2}te^{-2t}\sin t$ **12.** $te^{-2t}\sin \omega t$

Using (6) or (1), find $f(t)$ if $\mathcal{L}(f)$ equals

13. $\dfrac{4}{(s+1)^2}$ **14.** $\dfrac{8s}{(s^2+4)^2}$ **15.** $\dfrac{4s}{(s^2-4)^2}$ **16.** $\dfrac{2}{(s-a)^3}$
17. $\dfrac{s}{(s^2+1)^2}$ **18.** $\ln\dfrac{s+a}{s+b}$ **19.** $\ln\dfrac{s}{s-1}$ **20.** $\ln\dfrac{s^2+1}{(s-1)^2}$
21. arc cot (s/ω) **22.** arc cot $(s+1)$

23. Solve Probs. 2 and 3 by using the first shifting theorem.
24. Find $\mathcal{L}(t^n e^{at})$ by repeated application of (1), choosing $f(t) = e^{at}$.
25. Carry out the derivation in Example 3.

6.6 Convolution. Integral Equations

Another important general property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms $F(s)$ and $G(s)$ whose inverses $f(t)$ and $g(t)$ we know, and we would like to calculate the inverse of the product $H(s) = F(s)G(s)$ from those known inverses $f(t)$ and $g(t)$. This inverse $h(t)$ is written $(f * g)(t)$, which is a standard notation, and is called the **convolution** of f and g. How can we find h from f and g? This is stated in the following theorem. Since the situation and task just described arise quite often in applications, this theorem is of considerable practical importance.

$$(f * g) * v = f * (g * v) \qquad \text{(associative law)}$$

$$f * 0 = 0 * f = 0,$$

just as for numbers. But $f * 1 \neq f$ in general, as Example 2 shows. Another unusual property is that $(f * f)(t) \geqq 0$ may not hold, as we can see from Example 1.

Very useful applications of convolution occur in a natural way in the solution of differential equations, as we shall now discuss.

Differential Equations

From Sec. 6.2 we recall that the subsidiary equation of the differential equation

$$y'' + ay' + by = r(t) \tag{2}$$

has the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s) \tag{3}$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function. Hence for the solution $y(t)$ of (2) satisfying $y(0) = y'(0) = 0$ we have $Y = RQ$ in (3) and obtain from the convolution theorem the integral representation

$$y(t) = \int_0^t q(t - \tau)r(\tau)\, d\tau, \qquad q(t) = \mathcal{L}^{-1}(Q). \tag{4}$$

EXAMPLE 4 **Response of an undamped system to a single square wave**

We reconsider the model in Sec. 6.4, Example 2,

$$y'' + 2y = r(t), \qquad r(t) = 1 \text{ if } 0 < t < 1 \text{ and 0 otherwise}, \qquad y(0) = y'(0) = 0.$$

We solve it by the convolution technique in order to see how it works for ***inputs that act for some time only.***

Solution. We have $Q(s) = 1/(s^2 + 2)$, hence $q(t) = (\sin \sqrt{2}t)/\sqrt{2}$. Now we must be careful and remember that $r(t) = 1$ if $0 < t < 1$ but 0 if $t > 1$. Accordingly, in (4) we integrate from 0 to t when $t < 1$ but from 0 to 1 only when $t > 1$. Hence for $t < 1$ (we integrate over τ, so that the chain rule gives a -1),

$$y(t) = \frac{1}{\sqrt{2}} \int_0^t \sin \sqrt{2}(t - \tau)\, d\tau = \frac{1}{2} \cos \sqrt{2}(t - \tau)\Big|_0^t = \frac{1}{2}(1 - \cos \sqrt{2}t)$$

and for $t > 1$,

$$y(t) = \frac{1}{\sqrt{2}} \int_0^1 \sin \sqrt{2}(t - \tau)\, d\tau = \frac{1}{2}[\cos \sqrt{2}(t - 1) - \cos \sqrt{2}t].$$

Convolution also helps in solving certain **integral equations**, that is, equations in which the unknown function $y(t)$ appears under the integral (and perhaps also outside of it). This concerns only very special ones (those whose integral is of the form of a convolution), so that it suffices to consider a typical example and a handful of problems, but we do this because integral equations are practically important and often difficult to solve.

EXAMPLE 5 **Integral equation**

Solve the integral equation

$$y(t) = t + \int_0^t y(\tau) \sin (t - \tau)\, d\tau.$$

Solution. ***1st Step. Equation in terms of convolution.*** We see that the given equation can be written

$$y = t + y * \sin t.$$

2nd Step. Application of the convolution theorem. We write $Y = \mathcal{L}(y)$. By the convolution theorem,

$$Y(s) = \frac{1}{s^2} + Y(s)\frac{1}{s^2 + 1}.$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{s^2 + 1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}.$$

3rd Step. Taking the inverse transform. This gives the solution

$$y(t) = t + \tfrac{1}{6}t^3.$$

The reader may check this by substitution and evaluating the integral by repeated integration by parts (which will need patience).

Problem Set 6.6

Find the following convolutions. [*Hint.* In Probs. 7–10 use (11) in Appendix 3.]

1. $1 * 1$	**2.** $t * t^2$	**3.** $e^t * e^t$
4. $e^{kt} * e^{-kt}$	**5.** $t^2 * t^2$	**6.** $t * e^{at}$
7. $\sin \omega t * \cos \omega t$	**8.** $\sin \omega t * \sin \omega t$	**9.** $\sin t * \sin 2t$
10. $\cos \omega t * \cos \omega t$	**11.** $u(t - \pi) * \cos t$	**12.** $u(t - 1) * t$

Application of the Convolution Theorem. Find $h(t)$ by the convolution theorem if $H(s) = \mathcal{L}\{h(t)\}$ equals

13. $\dfrac{1}{(s-1)^2}$	**14.** $\dfrac{1}{s(s-1)}$	**15.** $\dfrac{1}{s^2(s-3)}$
16. $\dfrac{1}{s(s-2)^2}$	**17.** $\dfrac{1}{s(s^2+\omega^2)}$	**18.** $\dfrac{1}{s^2(s^2+\omega^2)}$

19. $\dfrac{s}{(s^2+\omega^2)^2}$ 20. $\dfrac{s^2}{(s^2+4)^2}$ 21. $\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

22. $\dfrac{s^2+a^2}{(s^2-a^2)^2}$ 23. $\dfrac{1}{(s+1)(s+2)}$ 24. $\dfrac{1}{(s^2+\omega^2)^2}$

General properties of convolution

25. Prove the commutative law $f * g = g * f$.
26. Prove the associative law $(f * g) * v = f * (g * v)$.
27. Prove the distributive law $f * (g_1 + g_2) = f * g_1 + f * g_2$.
28. **(Dirac's delta function)** Using the convolution theorem and treating Dirac's delta function (Sec. 6.4) as though it were an ordinary function, show that

$$(\delta * f)(t) = f(t).$$

29. Derive the formula in Prob. 28 by using f_k with $a = 0$ (Sec. 6.4) and applying the mean value theorem for integrals.
30. Using the convolution theorem, prove by induction that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{1}{(n-1)!}\,t^{n-1}e^{at}.$$

Application of convolution to initial value problems. Using the convolution theorem, solve:

31. $y'' + y = \sin 3t,\quad y(0) = 0,\quad y'(0) = 0$
32. $y'' + y = \sin t,\quad y(0) = 0,\quad y'(0) = 0$
33. $y'' + y = t,\quad y(0) = 0,\quad y'(0) = 0$
34. $y'' + 3y' + 2y = e^{-t},\quad y(0) = 0,\quad y'(0) = 0$
35. $y'' + 25y = 5.2e^{-t},\quad y(0) = 1.2,\quad y'(0) = -10.2$
36. $y'' + 2y = r(t)$, $r(t) = 1$ if $0 < t < 1$ and 0 if $t > 1$; $\quad y(0) = 0,\quad y'(0) = 0$
37. The gun barrel problem (Prob. 22 in Sec. 2.11).
38. $y'' + 3y' + 2y = r(t)$, $r(t) = 1$ if $0 < t < t_0$ and 0 if $t > t_0$; $\quad y(0) = 0$, $y'(0) = 0$
39. $y'' + 4y = u(t-1),\quad y(0) = 0,\quad y'(0) = 0$
40. $y'' + 3y' + 2y = 1 - u(t-1),\quad y(0) = 0,\quad y'(0) = 1$
41. $y'' + 9y = r(t)$, $r(t) = 8\sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $\quad y(0) = 0$, $y'(0) = 4$
42. $y'' - 5y' + 6y = r(t)$, $r(t) = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$; $\quad y(0) = 1$, $y'(0) = -2$
43. $y'' + 4y = r(t)$, $r(t) = 3\sin t$ if $0 < t < \pi$ and $-3\sin t$ if $t > \pi$; $y(0) = 0$, $y'(0) = 3$
44. $y'' + 3y' + 2y = r(t)$, $r(t) = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = y'(0) = 0$

Integral equations

Using Laplace transforms, solve:

45. $y(t) = 1 + \int_0^t y(\tau)\,d\tau$ 46. $y(t) = \sin 2t + \int_0^t y(\tau)\sin 2(t-\tau)\,d\tau$

47. $y(t) = 1 - \int_0^t (t-\tau)y(\tau)\,d\tau$ 48. $y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

49. $y(t) = te^t - 2e^t\int_0^t e^{-\tau}y(\tau)\,d\tau$

50. $y(t) = t + e^t - \int_0^t y(\tau)\cosh(t-\tau)\,d\tau$

6.7 Partial Fractions. Systems of Differential Equations

We have seen that partial fractions are needed to obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ of a problem from the solution $Y(s)$ of the subsidiary equation, because Y usually comes out as a quotient of two polynomials,

$$Y(s) = \frac{F(s)}{G(s)},$$

and for a partial fraction P, the inverse $\mathcal{L}^{-1}(P)$ is easy to get from a table and the first shifting theorem.

You may skip this section. Use your favorite method for partial fractions from calculus and any shortcuts you see. Call on this section for help only when you get stuck. We develop matters systematically in this order:

(Case 1) Unrepeated factor $s - a$.
(Case 2) Repeated factor $(s-a)^m$.
(Case 3) Complex factors $(s-a)(s-\bar{a})$.
(Case 4) Repeated complex factors $[(s-a)(s-\bar{a})]^2$.

Higher powers $[(s-a)(s-\bar{a})]^m$ are left aside because they are of minor practical interest. We discuss each case along with an example, and give the proofs for all four cases at the end. The formulas in this section are often called the **Heaviside expansion formulas.**

General assumption

$F(s)$ and $G(s)$ have real coefficients and no common factors. The degree of $F(s)$ is lower than that of $G(s)$.

Case 1. Unrepeated Factor $s - a$

To this factor corresponds in $Y = F/G$ a fraction

$$\frac{A}{s-a}. \tag{1a}$$

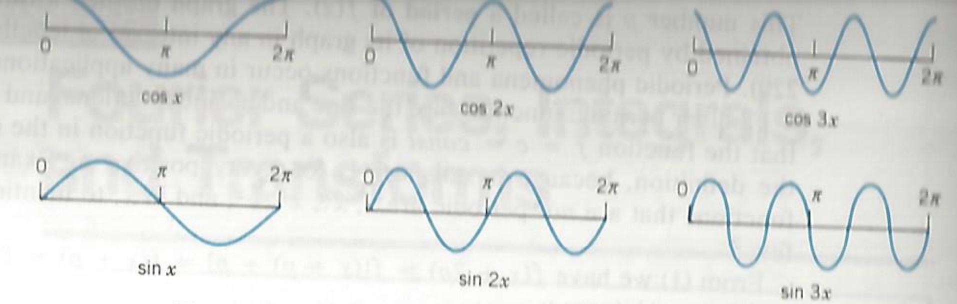


Fig. 230. Cosine and sine functions having the period 2π

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{4}$$

The set of functions (3) from which we have made up the series (4) is often called the **trigonometric system,** to have a short name for it.

We see that each term of the series (4) has the period 2π. Hence *if the series (4) converges, its sum will be a function of period 2π.*

Periodic functions that occur in practical problems are often rather complicated, and it is desirable to represent them in terms of simple periodic functions. We shall see that almost any periodic function $f(x)$ of period 2π that appears in applications—for example, in connection with vibrations—can be represented by a trigonometric series (which will then be called the *Fourier series* of f).

Problem Set 10.1

Find the smallest positive period p of the following functions.

1. $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi n x}{k}$, $\sin \frac{2\pi n x}{k}$

3. If $f(x)$ and $g(x)$ have period p, show that $h = af + bg$ (a, b constant) has the period p. Thus all functions of period p form a vector space.

4. If p is a period of $f(x)$, show that np, $n = 2, 3, \cdots$, is a period of $f(x)$.

5. Show that the function $f(x) = const$ is a periodic function of period p for every positive p.

6. If $f(x)$ is a periodic function of x of period p, show that $f(ax)$, $a \neq 0$, is a periodic function of x of period p/a, and $f(x/b)$, $b \neq 0$, is a periodic function of x of period bp. Verify these results for $f(x) = \cos x$, $a = b = 2$.

Sketch the following functions $f(x)$, which are assumed to be periodic of period 2π and, for $-\pi < x < \pi$, are given by the formulas

7. $f(x) = x$

8. $f(x) = x^2$

9. $f(x) = e^{|x|}$

10. $f(x) = |x|$

11. $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$

12. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 1 - x/\pi & \text{if } 0 < x < \pi \end{cases}$

13. $f(x) = \begin{cases} \pi + x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$

14. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ \cos x/2 & \text{if } 0 < x < \pi \end{cases}$

15. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$

16. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \sin x & \text{if } 0 < x < \pi \end{cases}$

Evaluate the following integrals where $n = 0, 1, 2, \cdots$. (These are typical examples of integrals that will be needed in our further work.)

17. $\int_0^{\pi} \sin nx \, dx$

18. $\int_{-\pi/2}^{0} \cos nx \, dx$

19. $\int_{-\pi/2}^{\pi/2} x \cos nx \, dx$

20. $\int_{-\pi}^{\pi} x \sin nx \, dx$

21. $\int_{-\pi}^{0} e^x \sin nx \, dx$

22. $\int_{-\pi/2}^{\pi/2} x \sin nx \, dx$

23. $\int_0^{\pi} e^x \cos nx \, dx$

24. $\int_{-\pi}^{\pi} x^2 \cos nx \, dx$

25. $\int_0^{\pi} x \sin nx \, dx$

10.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series (Sec. 10.1) whose coefficients are determined from $f(x)$ by certain formulas [the "Euler formulas" (6), below], which we shall derive first. Afterwards we shall take a look at the theory of Fourier series.

Euler Formulas for the Fourier Coefficients

Let us assume that $f(x)$ is a periodic function of period 2π that can be **represented** by a trigonometric series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \tag{1}$$

that is, we assume that this series converges and has $f(x)$ as its sum. Given such a function $f(x)$, we want to determine the coefficients a_n and b_n of the corresponding series (1).

We determine a_0. Integrating on both sides of (1) from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx.$$

Similarly, $|b_n| < 2M/n^2$ for all n. Hence the absolute value of each term of the Fourier series of $f(x)$ is at most equal to the corresponding term of the series

$$|a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$$

which is convergent. Hence that Fourier series converges and the proof is complete. (Readers already familiar with uniform convergence will see that, by the Weierstrass test in Sec. 14.6, under our present assumptions the Fourier series converges uniformly, and our derivation of (6) by integrating term by term is then justified by Theorem 3 of Sec. 14.6.)

The proof of convergence in the case of a piecewise continuous function $f(x)$ and the proof that under the assumptions in the theorem the Fourier series (7) with coefficients (6) represents $f(x)$ are substantially more complicated; see, for instance, Ref. [C14]. ■

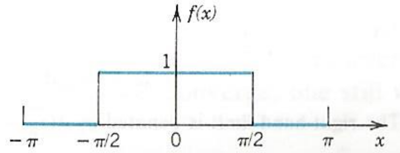
EXAMPLE 2 **Convergence at a jump as indicated in Theorem 1**

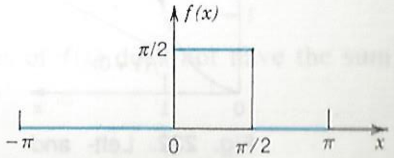
The square wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is k (Fig. 231), so that the average of these limits is 0. The Fourier series (8) of the square wave does indeed converge to this value when $x = 0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 1. ■

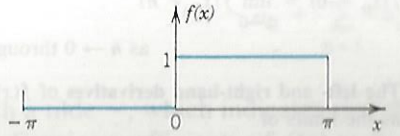
Summary. A Fourier series of a given function $f(x)$ of period 2π is a series of the form (7) with coefficients given by the Euler formulas (6). Theorem 1 gives conditions that are sufficient for this series to converge and at each x to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

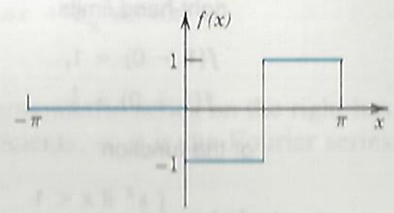
Problem Set 10.2

Find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the first three partial sums[8] where $f(x)$ equals

1.


2.


3.


4.


5. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

6. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$

7. $f(x) = x \ (-\pi < x < \pi)$

8. $f(x) = x \ (0 < x < 2\pi)$

9. $f(x) = x^2 \ (-\pi < x < \pi)$

10. $f(x) = x^2 \ (0 < x < 2\pi)$

11. $f(x) = x^3 \ (-\pi < x < \pi)$

12. $f(x) = x + |x| \ (-\pi < x < \pi)$

13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi \end{cases}$

15. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

16. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

17. Verify the last statement in Theorem 1 concerning the discontinuities for the function in Prob. 1.

18. Obtain the Fourier series in Prob. 3 from that in Prob. 1.

19. Show that if $f(x)$ has the Fourier coefficients a_n, b_n and $g(x)$ has the Fourier coefficients a_n^*, b_n^*, then $kf(x) + lg(x)$ has the Fourier coefficients $ka_n + la_n^*$, $kb_n + lb_n^*$.

20. Using Prob. 19, find the Fourier series in Prob. 2 from those in Probs. 3 and 4.

[8] That is, $a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$ for $N = 1, 2, 3$.

10.3 Functions of Any Period $p = 2L$

The functions considered so far had period 2π, whereas most periodic functions in applications will have other periods. But we show that the transition from functions of period $p = 2\pi$ to those of period[9] $p = 2L$ is quite simple, basically a stretch of scale on the axis.

If a function $f(x)$ of period $p = 2L$ has a **Fourier series,** we claim that this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \tag{1}$$

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas**

(2)

(a) $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

(b) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots,$$

(c) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots.$$

[9] This notation is practical since in applications, L will be the length of a vibrating string (Sec. 11.2), of a rod in heat conduction (Sec. 11.5), etc.

Problem Set 10.3

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the first three partial sums.

1. $f(x) = -1 \quad (-1 < x < 0), \quad f(x) = 1 \quad (0 < x < 1), \quad p = 2L = 2$
2. $f(x) = 0 \quad (-1 < x < 1), \quad f(x) = 1 \quad (1 < x < 3), \quad p = 2L = 4$
3. $f(x) = 0 \quad (-2 < x < 0), \quad f(x) = 2 \quad (0 < x < 2), \quad p = 2L = 4$
4. $f(x) = x \quad (-1 < x < 1), \quad p = 2L = 2$
5. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
6. $f(x) = 2|x| \quad (-2 < x < 2), \quad p = 2L = 4$
7. $f(x) = 0, \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1), \quad p = 2L = 2$
8. $f(x) = x \quad (0 < x < 1), \quad f(x) = 1 - x \quad (1 < x < 2), \quad p = 2L = 2$
9. $f(x) = -1 \quad (-1 < x < 0), \quad f(x) = 2x \quad (0 < x < 1), \quad p = 2L = 2$
10. $f(x) = \frac{1}{2} + x \quad (-\frac{1}{2} < x < 0), \quad f(x) = \frac{1}{2} - x \quad (0 < x < \frac{1}{2}), \quad p = 2L = 1$
11. $f(x) = 3x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
12. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2L = 2$
13. $f(x) = \pi \sin \pi x \quad (0 < x < 1), \quad p = 2L = 1$
14. $f(x) = x^2/4 \quad (0 < x < 2), \quad p = 2L = 2$

15. Obtain the Fourier series in Prob. 1 directly from that in Example 1, Sec. 10.2.
16. Obtain the Fourier series in Prob. 11 directly from that in Prob. 9, Sec. 10.2.
17. Obtain the Fourier series in Prob. 3 directly from that in Example 1.
18. Find the Fourier series of the periodic function that is obtained by passing the voltage $v(t) = k \cos 100\pi t$ through a half-wave rectifier.
19. Show that each term in (1) has the period $p = 2L$.
20. Show that in (2) the interval of integration may be replaced by any interval of length $p = 2L$.

10.4 Even and Odd Functions

The function in Example 1 of the last section was odd and had only sine terms in its Fourier series, no cosine terms. This is typical. In fact, unnecessary work (and corresponding sources of errors) in determining Fourier coefficients can be avoided if a function is odd or even.

We first remember that a function $y = g(x)$ is **even** if

$$g(-x) = g(x) \qquad \text{for all } x.$$

The graph of such a function is symmetric with respect to the y-axis (Fig. 235). A function $h(x)$ is **odd** if

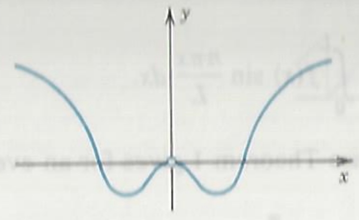


Fig. 235. Even function

Fig. 236. Odd function

If $g(x)$ is an even function, then

$$(1) \qquad \int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx \qquad (g \text{ even}).$$

If $h(x)$ is an odd function, then

$$(2) \qquad \int_{-L}^{L} h(x)\,dx = 0 \qquad (h \text{ odd}).$$

Formulas (1) and (2) are obvious from the graphs of g and h, and we leave the formal proofs to the student.

The product $q = gh$ of an even function g and an odd function h is odd, because

$$q(-x) = g(-x)h(-x) = g(x)[-h(x)] = -q(x).$$

Hence if $f(x)$ is even, then the integrand $f \sin (n\pi x/L)$ in (2c), Sec. 10.3, is odd, and $b_n = 0$. Similarly, if $f(x)$ is odd, then $f \cos (n\pi x/L)$ in (2b), Sec. 10.3, is odd, and $a_n = 0$. From this and (1) we obtain the following theorem.

Theorem 1 (Fourier series of even and odd functions)
The Fourier series of an even function of period 2L is a **"Fourier cosine series"**

$$(3) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$

with coefficients

$$(4) \qquad a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

The Fourier series of an odd function of period 2L is a **"Fourier sine series"**

Problem Set 10.4

Are the following functions odd, even, or neither odd nor even?

1. $|x^3|$, $x \cos nx$, $x^2 \cos nx$, $\cosh x$, $\sinh x$, $\sin x + \cos x$, $x|x|$

2. $x + x^2$, $|x|$, e^x, e^{x^2}, $\sin^2 x$, $x \sin x$, $\ln x$, $x \cos x$, $e^{-|x|}$

Are the following functions $f(x)$, which are assumed to be periodic, of period 2π, even, odd or neither even nor odd?

3. $f(x) = x^3 \quad (-\pi < x < \pi)$

4. $f(x) = x^4 \quad (0 < x < 2\pi)$

5. $f(x) = x|x| \quad (-\pi < x < \pi)$

6. $f(x) = e^{-|x|} \quad (-\pi < x < \pi)$

7. $f(x) = |\sin x| \quad (-\pi < x < \pi)$

8. $x^3 - x$

9. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0 & \text{if } 1 < x < 2\pi - 1 \\ x & \text{if } -1 < x < 1 \end{cases}$

11. $f(x) = \begin{cases} \sinh x & \text{if } -\pi < x < 0 \\ -\cosh x & \text{if } 0 < x < \pi \end{cases}$

12. $f(x) = \begin{cases} \cos^2 2x & \text{if } -\pi < x < 0 \\ \sin^2 2x & \text{if } 0 < x < \pi \end{cases}$

Represent the following functions as the sum of an even and an odd function.

13. $1/(1 - x)$ **14.** $1/(1 - x)^2$ **15.** e^{kx} **16.** $x/(x + 1)$

17. Prove Theorem 2.

18. Find all functions that are both even and odd.

19. Show that the familiar identity $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as a Fourier series expansion, and the same holds for the identity $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$.

Prove:

20. The sum and the product of even functions are even functions.

21. The sum of odd functions is odd. The product of two odd functions is even.

22. If $f(x)$ is odd, then $|f(x)|$ and $f^2(x)$ are even functions.

23. If $f(x)$ is even, then $|f(x)|$, $f^2(x)$, and $f^3(x)$ are even functions.

24. If $g(x)$ is defined for all x, then the function $p(x) = [g(x) + g(-x)]/2$ is even and the function $q(x) = [g(x) - g(-x)]/2$ is odd.

Find the Fourier series of the following functions, which are assumed to have the period 2π. *Hint.* Use that some of these functions are even or odd.

25. $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

26. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } \pi < x < 2\pi \end{cases}$

27. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

28. $f(x) = \begin{cases} -x & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi \end{cases}$

29. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

30. $f(x) = \begin{cases} -x^2 & \text{if } -\pi < x < 0 \\ x^2 & \text{if } 0 < x < \pi \end{cases}$

31. $f(x) = x^2/4 \quad (-\pi < x < \pi)$

32. $f(x) = x(\pi^2 - x^2) \quad (-\pi < x < \pi)$

Show that

33. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$ (Use Prob. 25.)

34. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (Use Prob. 31.)

35. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{\pi^2}{12}$ (Use Prob. 31.)

10.5 Half-Range Expansions

In various applications there is a practical need to use Fourier series in connection with functions $f(x)$ that are given on some interval only, say, $0 \leqq x \leqq L$, as in Fig. 239a. Typical cases follow in the next chapter (Secs. 11.3 and 11.5). We could extend $f(x)$ periodically with period L and then represent the extended function by a Fourier series, which in general would involve both cosine *and* sine terms. We can do better and always get a cosine series by first extending $f(x)$ from $0 \leqq x \leqq L$ as an *even* function on the range (the interval) $-L \leqq x \leqq L$, as in Fig. 239b, and then extend this new function as a periodic function of period $2L$ and, since it is even, represent it by a Fourier cosine series. Or we can extend $f(x)$ from $0 \leqq x \leqq L$ as an

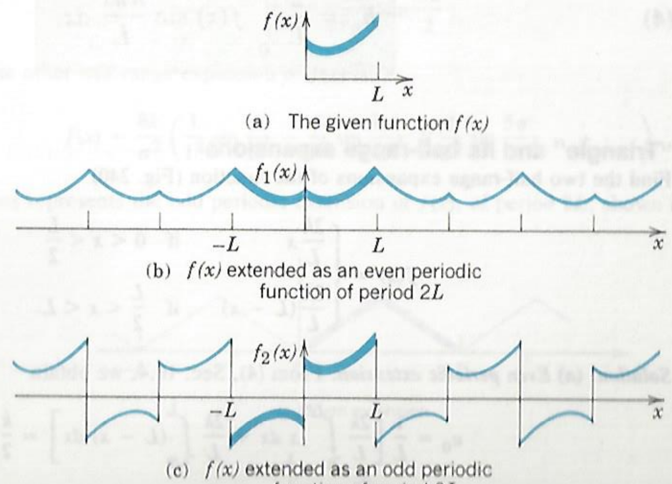


Fig. 239. **(a)** Function $f(x)$ given on an interval $0 \leqq x \leqq L$, **(b)** its even extension to the full "range" (interval) $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis, **(c)** its odd extension to $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

Problem Set 10.5

Represent the following functions $f(x)$ by a Fourier sine series and sketch the corresponding periodic extension of $f(x)$.

1. $f(x) = k \quad (0 < x < L)$
2. $f(x) = kx \quad (0 < x < L)$
3. $f(x) = x^2 \quad (0 < x < L)$
4. $f(x) = 1 - (2/L)x \quad (0 < x < L)$
5. $f(x) = L - x \quad (0 < x < L)$
6. $f(x) = x^3 \quad (0 < x < L)$
7. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi/2 \\ \pi/2 & \text{if } \pi/2 < x < \pi \end{cases}$
8. $f(x) = \begin{cases} \pi/2 & \text{if } 0 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < \pi \end{cases}$
9. $f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L - x & \text{if } L/2 < x < L \end{cases}$
10. $f(x) = \begin{cases} x & \text{if } 0 < x < L/2 \\ L/2 & \text{if } L/2 < x < L \end{cases}$

Represent the following functions $f(x)$ by a Fourier cosine series and sketch the corresponding periodic extension of $f(x)$.

11. $f(x) = x \quad (0 < x < L)$
12. $f(x) = 1 \quad (0 < x < L)$
13. $f(x) = x^2 \quad (0 < x < L)$
14. $f(x) = \sin^2 3x \quad (0 < x < \pi)$
15. $f(x) = \begin{cases} 0 & \text{if } 0 < x < L/2 \\ 1 & \text{if } L/2 < x < L \end{cases}$
16. $f(x) = \begin{cases} 1 & \text{if } 0 < x < L/2 \\ 0 & \text{if } L/2 < x < L \end{cases}$
17. $f(x) = e^x \quad (0 < x < L)$
18. $f(x) = x^3 \quad (0 < x < L)$
19. $f(x) = \sin \dfrac{\pi x}{L} \quad (0 < x < L)$
20. $f(x) = \dfrac{\pi}{4} \sin \dfrac{\pi x}{2L} \quad (0 < x < L)$

10.6 Complex Fourier Series *Optional*

In this optional section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

can be written in complex form, which sometimes simplifies calculations (see Example 1, below). This is done by the Euler formula (8), Sec. 2.3,

$$e^{it} = \cos t + i \sin t$$

and its companion

$$e^{-it} = \cos t - i \sin t$$

[obtained from $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$] with $t = nx$, that is,

$$e^{inx} = \cos nx + i \sin nx, \tag{2}$$

$$e^{-inx} = \cos nx - i \sin nx. \tag{3}$$

By addition and division by 2 we get

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}). \tag{4}$$

Subtraction and division by $2i$ gives

$$\sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}). \tag{5}$$

From this, using $1/i = -i$, we have in (1)

$$a_n \cos nx + b_n \sin nx = \frac{1}{2} a_n(e^{inx} + e^{-inx}) + \frac{1}{2i} b_n(e^{inx} - e^{-inx})$$
$$= \tfrac{1}{2}(a_n - ib_n)e^{inx} + \tfrac{1}{2}(a_n + ib_n)e^{-inx}.$$

With this, (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}) \tag{6}$$

where $c_0 = a_0$, and by (1)–(3) and the Euler formulas (6), Sec. 10.2,

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,$$
$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx, \qquad n = 1, 2, \cdots. \tag{7}$$

Finally, if we introduce the notation $k_n = c_{-n}$, we obtain from (6) and (7)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{8}$$

This is the so-called **complex form of the Fourier series** or, more briefly, the **complex Fourier series**, of $f(x)$, and the c_n are called the **complex Fourier coefficients** of $f(x)$.

10.8 Approximation by Trigonometric Polynomials

A main field of application of Fourier series is in differential equations, as we have said. Another area of practical interest in which Fourier series play a major role is the approximation of functions by simpler functions, known as **approximation theory,** as we shall now explain.

Let $f(x)$ be a function of period 2π that can be represented by a Fourier series. Then the Nth partial sum of this series is an approximation to $f(x)$:

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx). \tag{1}$$

It is natural to ask whether (1) is the "best" approximation to f by a **trigonometric polynomial** *of degree* N (N fixed), that is, a function of the form

$$F(x) = a_0 + \sum_{n=1}^{N} (\alpha_n \cos nx + \beta_n \sin nx), \tag{2}$$

where "best" means that the "error" of the approximation is minimum.

Of course, we must first define what we mean by the error E of such an approximation. We want to choose a definition that measures the goodness of agreement between f and F *on the whole interval* $-\pi \leqq x \leqq \pi$. Obviously, the maximum of $|f - F|$ is not suitable for that purpose: in Fig. 246, the function F is a good approximation to f, but $|f - F|$ is large near x_0. We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx. \tag{3}$$

This is called the **total square error** of F relative to the function f on the interval $-\pi \leqq x \leqq \pi$. Clearly, $E \geqq 0$.

N being fixed, we want to determine the coefficients in (2) such that E is minimum. Since $(f - F)^2 = f^2 - 2fF + F^2$, we have

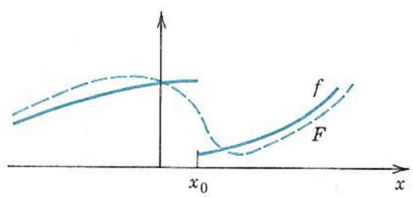


Fig. 246. Error of approximation

(4) $$E = \int_{-\pi}^{\pi} f^2\,dx - 2\int_{-\pi}^{\pi} fF\,dx + \int_{-\pi}^{\pi} F^2\,dx.$$

By inserting (2) into the last integral and evaluating the occurring integrals as in Sec. 10.2 we see that all terms $\cos^2 nx$ and $\sin^2 nx$ $(n \geqq 1)$ have integral π and all "mixed terms" $(\cos nx)(\sin mx)$ have integral zero. Thus

$$\int_{-\pi}^{\pi} F^2\,dx = \pi(2\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_N^2 + \beta_1^2 + \cdots + \beta_N^2).$$

By inserting (2) into the second integral in (4) we see that the occurring integrals are those in the Euler formulas (6), Sec. 10.2, and we thus obtain

$$\int_{-\pi}^{\pi} fF\,dx = \pi(2\alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_N a_N + \beta_1 b_1 + \cdots + \beta_N b_N).$$

With these expressions (4) becomes

(5) $$\begin{aligned} E = &\int_{-\pi}^{\pi} f^2\,dx \\ &- 2\pi\left[2\alpha_0 a_0 + \sum_{n=1}^{N} (\alpha_n a_n + \beta_n b_n)\right] \\ &+ \pi\left[2\alpha_0^2 + \sum_{n=1}^{N} (\alpha_n^2 + \beta_n^2)\right]. \end{aligned}$$

If we take $\alpha_n = a_n$ and $\beta_n = b_n$ in (2), then from (5) we see that the square error corresponding to this choice of the coefficients of F is given by

(6) $$E^* = \int_{-\pi}^{\pi} f^2\,dx - \pi\left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right].$$

By subtracting (6) from (5) we obtain

$$E - E^* = \pi\left\{2(\alpha_0 - a_0)^2 + \sum_{n=1}^{N} [(\alpha_n - a_n)^2 + (\beta_n - b_n)^2]\right\}.$$

Since the sum of squares of real numbers on the right cannot be negative,

$$E - E^* \geqq 0, \quad \text{thus} \quad E \geqq E^*,$$

and $E = E^*$ if and only if $\alpha_0 = a_0, \cdots, \beta_N = b_N$. This proves

Theorem 1 **(Minimum square error)**
The total square error of F in (2) (with fixed N) relative to f on the interval $-\pi \leqq x \leqq \pi$ is minimum if and only if the coefficients of F in (2) are the Fourier coefficients of f. This minimum value is given by (6).

From (6) we see that E^* cannot increase as N increases, but may decrease. Hence, *with increasing N, the partial sums of the Fourier series of f yield better and better approximations to f*, considered from the viewpoint of the square error.

Since $E^* \geqq 0$ and (6) holds for every N, we obtain from (6) the important **Bessel inequality**[10]

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx \tag{7}$$

for the Fourier coefficients of any function f for which the integral on the right exists.

It can be shown (see [C14]) that for such a function f, **Parseval's theorem** holds, that is, formula (7) holds with the equality sign, so that it becomes **"Parseval's identity"**[11]

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx. \tag{8}$$

EXAMPLE 1 **Square error for the sawtooth wave**

Compute the total square error of F with $N = 3$ relative to

$$f(x) = x + \pi \quad (-\pi < x < \pi) \qquad \text{(Fig. 238a, Sec. 10.4)}$$

on the interval $-\pi \leqq x \leqq \pi$.

Solution. $F(x) = \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$ by Example 2, Sec. 10.4. From this and (6),

$$E^* = \int_{-\pi}^{\pi} (x + \pi)^2\,dx - \pi[2\pi^2 + 2^2 + 1^2 + (\tfrac{2}{3})^2],$$

hence

$$E^* = \tfrac{8}{3}\pi^3 - \pi(2\pi^2 + \tfrac{49}{9}) \approx 3.567.$$

$F = S_3$ is shown in Fig. 238b, and although $|f(x) - F(x)|$ is large at $x = \pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval. ■

This brings to an end our discussion of Fourier series, which has emphasized the practical aspects of these series, as needed in applications. In the last four sections of this chapter we show how ideas and techniques in Fourier series can be extended to nonperiodic functions.

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[10] See footnote 12 in Sec. 5.5.
[11] MARC ANTOINE PARSEVAL (1755—1836), French mathematician. A physical interpretation of the identity follows in the next section.

We denote the integral by I and show that it equals $\sqrt{\pi/a}$. For this we use $\sqrt{a}x + iw/2\sqrt{a} = v$ as a new variable of integration. Then $dx = dv/\sqrt{a}$, so that

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2}\, dv.$$

We now get the result by the following trick. We square the integral, convert it to a double integral, and use polar coordinates $r = \sqrt{u^2 + v^2}$ and θ. Since $du\, dv = r\, dr\, d\theta$, we get

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv = \frac{1}{a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, du\, dv$$

$$= \frac{1}{a} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2}\, r\, dr\, d\theta = \frac{2\pi}{a}\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{\infty} = \frac{\pi}{a}.$$

Hence $I = \sqrt{\pi/a}$. From this and the first formula in this solution,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{4a}\right) \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}\, e^{-w^2/4a}.$$

This agrees with formula 9 in Table III, Sec. 10.12. ■

Physical Interpretation: Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a **spectral representation.** This name is suggested by optics, where light is such a superposition of colors (frequencies). In (7), the **"spectral density"** $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$ (Δw small, fixed). We claim that in connection with vibrations, the integral

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw$$

can be interpreted as the **total energy** of the physical system; hence an integral of $|\hat{f}(w)|^2$ from a to b gives the contribution of the frequencies w between a and b to the total energy.

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring, Sec. 2.5)

$$my'' + ky = 0,$$

denoting time t by x. Multiplication by y' and integration gives

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = E_0 = const,$$

where $v = y'$ is the velocity, the first term is the kinetic energy, the second the potential energy, and E_0 the total energy of the system. Now a general solution is [use (4), (5), Sec. 10.6]

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x}, \quad w_0^2 = k/m,$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$. Since $mw_0^2 = k$ and $(iw_0)^2 = -w_0^2$, we get by straightforward calculation and simplification

$$E_0 = \tfrac{1}{2}m(c_1 iw_0 e^{iw_0 x} - c_{-1} iw_0 e^{-iw_0 x})^2 + \tfrac{1}{2}k(c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x})^2$$
$$= 2kc_1 c_{-1} = 2k|c_1|^2.$$

Hence *the energy is proportional to the square of the amplitude* $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (8), Sec. 10.6. In this case we have a **"discrete spectrum"** (or **"point spectrum"**) consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

Finally, a system whose solution can be represented by a Fourier integral (7) leads to the above integral for the energy, as is plausible from the cases just discussed.

Linearity. Fourier Transform of Derivatives

New transforms can be obtained from given ones by

Theorem 1 **(Linearity of the Fourier transform)**
The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g). \tag{8}$$

Proof. This is true since integration is a linear operation, so that (6) gives

$$\mathcal{F}\{af(x) + bg(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-iwx}\, dx$$
$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx}\, dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-iwx}\, dx$$
$$= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$ ∎

In the application of the Fourier transform to differential equations, the key property is that differentiation of functions corresponds to multiplication of transforms by iw:

work and avoiding errors. The simplifications follow immediately from our formulas just obtained.

Indeed, if $f(x)$ is an even function, then $B(w) = 0$ in (4) and

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\, dv, \tag{10}$$

and the Fourier integral (5) reduces to the so-called **Fourier cosine integral**

$$f(x) = \int_0^\infty A(w)\cos wx\, dw \qquad (f \text{ even}). \tag{11}$$

Similarly, if $f(x)$ is odd, then in (4) we have $A(w) = 0$ and

$$B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin wv\, dv, \tag{12}$$

and the Fourier integral (5) reduces to the so-called **Fourier sine integral**

$$f(x) = \int_0^\infty B(w)\sin wx\, dw \qquad (f \text{ odd}). \tag{13}$$

Evaluation of Integrals

Fourier integral representations may also be used for evaluating integrals. We illustrate this method with a typical example.

EXAMPLE 3 Laplace integrals

Find the Fourier cosine and sine integrals of

$$f(x) = e^{-kx} \qquad (x > 0,\ k > 0).$$

Solution. (a) From (10) we have

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv.$$

Now, by integration by parts,

$$\int e^{-kv}\cos wv\, dv = -\frac{k}{k^2+w^2}e^{-kv}\left(-\frac{w}{k}\sin wv + \cos wv\right).$$

If $v = 0$, the expression on the right equals $-k/(k^2 + w^2)$; if v approaches infinity, it approaches zero because of the exponential factor. Thus

$$A(w) = \frac{2k/\pi}{k^2+w^2}. \tag{14}$$

From this representation we see that

$$\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw = \frac{\pi}{2k}e^{-kx} \qquad (x > 0,\ k > 0). \tag{15}$$

(b) Similarly, from (12) we have

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\, dv.$$

By integration by parts,

$$\int e^{-kv}\sin wv\, dv = -\frac{w}{k^2+w^2}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right).$$

This equals $-w/(k^2 + w^2)$ if $v = 0$, and approaches 0 as $v \to \infty$. Thus

$$B(w) = \frac{2w/\pi}{k^2+w^2}. \tag{16}$$

From (13) we thus obtain the Fourier sine integral representation

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\sin wx}{k^2+w^2}\, dw.$$

From this we see that

$$\int_0^\infty \frac{w\sin wx}{k^2+w^2}\, dw = \frac{\pi}{2}e^{-kx} \qquad (x > 0,\ k > 0). \tag{17}$$

The integrals (15) and (17) are the so-called **Laplace integrals.** ■

Problem Set 10.9

Using the Fourier integral representation, show that

1. $\displaystyle\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$ [Use (5).]

2. $\displaystyle\int_0^\infty \frac{w^3\sin xw}{w^4+4}\, dw = \frac{\pi}{2}e^{-x}\cos x \quad \text{if } x > 0$ [Use (13).]

3. $\displaystyle\int_0^\infty \frac{\cos xw}{1+w^2}\, dw = \frac{\pi}{2}e^{-x} \quad \text{if } x > 0$ [Use (11).]

4. $\displaystyle\int_0^\infty \frac{\sin w\cos xw}{w}\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$ [Use (11).]

5. $\displaystyle\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leqq x \leqq \pi \\ 0 & \text{if } x > \pi \end{cases}$ [Use (13).]

6. $\displaystyle\int_0^\infty \frac{1-\cos \pi w}{w}\sin xw\,dw = \begin{cases} \frac{\pi}{2} & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$ [Use (13).]

7. $\displaystyle\int_0^\infty \frac{\cos(\pi w/2)\cos xw}{1-w^2}\,dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if } |x| < \frac{\pi}{2} \\ 0 & \text{if } |x| > \frac{\pi}{2} \end{cases}$ [Use (11).]

Represent the following functions $f(x)$ in the form (11).

8. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

9. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

10. $f(x) = \begin{cases} x/2 & \text{if } 0 < x < 1 \\ 1 - x/2 & \text{if } 1 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$

11. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

12. $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

13. $f(x) = \dfrac{1}{1+x^2}$ [see (15)]

14. $f(x) = e^{-x} + e^{-2x}$ $(x > 0)$

If $f(x)$ has the representation (11), show that

15. $f(ax) = \dfrac{1}{a}\displaystyle\int_0^\infty A\left(\frac{w}{a}\right)\cos xw\,dw$ $(a > 0)$

16. $xf(x) = \displaystyle\int_0^\infty B^*(w)\sin xw\,dw,\quad B^* = -\frac{dA}{dw},$ A as in (10)

17. $x^2 f(x) = \displaystyle\int_0^\infty A^*(w)\cos xw\,dw,\quad A^* = -\frac{d^2A}{dw^2}$

18. Solve Prob. 9 by applying the formula in Prob. 17 to the result of Prob. 8.

19. Verify the formula in Prob. 16 for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

20. Show that $f(x) = 1$ $(0 < x < \infty)$ cannot be represented by a Fourier integral.

10.10 Fourier Cosine and Sine Transforms

equations, and they often also help in handling and applying special functions. The **Laplace transform** (Chap. 6) is of this kind and is by far the most important integral transform in engineering. From the viewpoint of applications, the next in order of importance are perhaps the **Fourier transforms,** although these are somewhat more difficult to handle than the Laplace transform. We shall see that they can be obtained from the Fourier integral representations in Sec. 10.9. In this section we consider two of them, called the *Fourier cosine* and *sine transforms,* which are real, and in the next section a third one, which is complex.

Fourier Cosine Transforms

For an ***even*** function $f(x)$, the Fourier integral is the Fourier cosine integral

(1) (a) $f(x) = \displaystyle\int_0^\infty A(w)\cos wx\,dw,$ where (b) $A(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(v)\cos wv\,dv$

[see (10), (11), Sec. 10.9]. We now set $A(w) = \sqrt{2/\pi}\,\hat{f}_c(w)$, where c suggests "cosine." Then from (1b), writing $v = x$, we have

(2) $$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

and from (1a),

(3) $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw.$$

Attention! In (2) we integrate with respect to x and in (3) with respect to w. Formula (2) gives from $f(x)$ a new function $\hat{f}_c(w)$, called the **Fourier cosine transform** of $f(x)$. Formula (3) gives us back $f(x)$ from $\hat{f}_c(w)$, and we therefore call $f(x)$ the **inverse Fourier cosine transform** of $\hat{f}_c(w)$.

The process of obtaining the transform $\hat{f}_c$ from a given f is also called the **Fourier cosine transform** or the *Fourier cosine transform method.*

Fourier Sine Transforms

Similarly, for an ***odd*** function $f(x)$, the Fourier integral is the Fourier sine integral [see (12), (13), Sec. 10.9]

(4) (a) $f(x) = \displaystyle\int_0^\infty B(w)\sin wx\,dw,$ where (b) $B(w) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(v)\sin wv\,dv,$

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \sin wx \Big|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right]$$

$$= 0 - w\mathcal{F}_c\{f(x)\}.$$

Formula (8a) with f' instead of f gives

$$\mathcal{F}_c\{f''(x)\} = w\mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}} f'(0);$$

hence by (8b),

(9a)
$$\mathcal{F}_c\{f''(x)\} = -w^2\mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0).$$

Similarly,

(9b)
$$\mathcal{F}_s\{f''(x)\} = -w^2\mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} wf(0).$$

An application of (9) to differential equations will be given in Sec. 11.14. For the time being, we show how (9) can be used to derive transforms.

EXAMPLE 3 **An application of the operational formula (9)**

Find the Fourier cosine transform of $f(x) = e^{-ax}$, where $a > 0$.

Solution. By differentiation, $(e^{-ax})'' = a^2e^{-ax}$; thus $a^2f(x) = f''(x)$. From this and (9a),

$$a^2\mathcal{F}_c(f) = \mathcal{F}_c(f'') = -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -w^2\mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}.$$

Hence $(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}$. The answer is (see Table I, Sec. 10.12)

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2} \right) \qquad (a > 0).$$

Tables of Fourier cosine and sine transforms are included in Sec. 10.12. For more extensive tables, see Ref. [C4] in Appendix 1.

Problem Set 10.10

1. Let $f(x) = -1$ if $0 < x < 1$, $f(x) = 1$ if $1 < x < 2$. Find $\hat{f}_c(w)$.
2. Derive $f(x)$ in Prob. 1 from the answer to Prob. 1. *Hint.* Use Prob. 4 in Sec. 10.9.
3. Find the Fourier cosine transform of $f(x) = x$ if $0 < x < a$, $f(x) = 0$ if $x > a$.
4. Find $\mathcal{F}_s(e^{-ax})$, $a > 0$, by integration.
5. Derive formula 3 in Table I of Sec. 10.12 by integration.
6. Obtain the answer to Prob. 4 from (9b).
7. Obtain $\mathcal{F}_c(1/(1 + x^2))$ from Prob. 3 in Sec. 10.9.
8. Obtain the inverse Fourier cosine transform of e^{-w}.
9. Find the Fourier sine transform of $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.
10. Find the Fourier cosine transform of the function in Prob. 9.
11. Obtain $\mathcal{F}_s(x^{-1} - x^{-1} \cos \pi x)$. *Hint.* Use Prob. 6, Sec. 10.9, with w and x interchanged.
12. Obtain formula 10 in Table I of Sec. 10.12 with $a = 1$ from Example 2 in Sec. 10.9.
13. Find $\mathcal{F}_c\{(\cos \pi x/2)/(1 - x^2)\}$. *Hint.* Use Prob. 7, Sec. 10.9.
14. Using (8b), obtain $\mathcal{F}_s(xe^{-x^2/2})$ from a suitable formula in Table I, Sec. 10.12.
15. Find $\mathcal{F}_s(e^{-x})$ from (8a) and formula 3 of Table I, Sec. 10.12.
16. Using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, obtain formula 2 in Table II, Sec. 10.12, from formula 4 in that table.
17. Let $f(x) = x^3/(x^4 + 4)$. Find $\hat{f}_s(w)$ for $w > 0$. *Hint.* Use Prob. 2 in Sec. 10.9.
18. Show that $f(x) = 1$ has no Fourier cosine or sine transform.
19. Do the Fourier cosine and sine transforms of $f(x) = e^x$ exist?
20. Does the Fourier cosine transform of $(\cos x)/x$ exist? Of $(\sin x)/x$?

10.11 Fourier Transform

The previous section concerned two transforms obtained from the Fourier cosine and sine integrals in Sec. 10.9. We now consider a third transform, the *Fourier transform*, which is obtained from the Fourier integral in complex form. (For a motivation of this transform, see the beginning of Sec. 10.10.) We therefore consider first the complex form of the Fourier integral.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 10.9]

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] \, dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv, \qquad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv.$$

Substituting A and B into the

We denote the integral by I and show that it equals $\sqrt{\pi/a}$. For this we use $\sqrt{a}x + iw/2\sqrt{a} = v$ as a new variable of integration. Then $dx = dv/\sqrt{a}$, so that

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-v^2}\, dv.$$

We now get the result by the following trick. We square the integral, convert it to a double integral, and use polar coordinates $r = \sqrt{u^2 + v^2}$ and θ. Since $du\, dv = r\, dr\, d\theta$, we get

$$I^2 = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv = \frac{1}{a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, du\, dv$$

$$= \frac{1}{a} \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta = \frac{2\pi}{a} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^{\infty} = \frac{\pi}{a}.$$

Hence $I = \sqrt{\pi/a}$. From this and the first formula in this solution,

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{4a} \right) \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}} e^{-w^2/4a}.$$

This agrees with formula 9 in Table III, Sec. 10.12. ■

Physical Interpretation: Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a **spectral representation.** This name is suggested by optics, where light is such a superposition of colors (frequencies). In (7), the **"spectral density"** $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$ (Δw small, fixed). We claim that in connection with vibrations, the integral

$$\int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw$$

can be interpreted as the **total energy** of the physical system; hence an integral of $|\hat{f}(w)|^2$ from a to b gives the contribution of the frequencies w between a and b to the total energy.

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring, Sec. 2.5)

$$my'' + ky = 0,$$

denoting time t by x. Multiplication by y' and integration gives

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2 = E_0 = const,$$

where $v = y'$ is the velocity, the first term is the kinetic energy, the second the potential energy, and E_0 the total energy of the system. Now a general solution is [use (4), (5), Sec. 10.6]

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x}, \quad w_0^2 = k/m,$$

where $c_1 = (a_1 - ib_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + ib_1)/2$. Since $mw_0^2 = k$ and $(iw_0)^2 = -w_0^2$, we get by straightforward calculation and simplification

$$E_0 = \tfrac{1}{2}m(c_1 iw_0 e^{iw_0 x} - c_{-1} iw_0 e^{-iw_0 x})^2 + \tfrac{1}{2}k(c_1 e^{iw_0 x} + c_{-1} e^{-iw_0 x})^2$$

$$= 2kc_1 c_{-1} = 2k|c_1|^2.$$

Hence *the energy is proportional to the square of the amplitude* $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (8), Sec. 10.6. In this case we have a **"discrete spectrum"** (or **"point spectrum"**) consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

Finally, a system whose solution can be represented by a Fourier integral (7) leads to the above integral for the energy, as is plausible from the cases just discussed.

Linearity. Fourier Transform of Derivatives

New transforms can be obtained from given ones by

Theorem 1 **(Linearity of the Fourier transform)**
The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and b,

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g). \tag{8}$$

Proof. This is true since integration is a linear operation, so that (6) gives

$$\mathcal{F}\{af(x) + bg(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-iwx}\, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-iwx}\, dx$$

$$= a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}.$$ ■

In the application of the Fourier transform to differential equations, the key property is that differentiation of functions corresponds to multiplication of transforms by iw:

Problem Set 10.11

Find the Fourier transforms of the following functions $f(x)$ (without using Table III, Sec. 10.12).

1. $f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

2. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$

3. $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{x} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

5. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} xe^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

7. $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } \ 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

8. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$

9. Derive formula 1 in Table III, Sec. 10.12.

10. Using the answer $\hat{f}(w)$ to Prob. 9, obtain $f(x)$ from (7). *Hint.* Use (7*) in Sec. 10.9.

11. Obtain formula 1 in Table III, Sec. 10.12, from formula 2 in that table.

12. Solve Prob. 6 by (9), using the answer to Prob. 1.

13. **(Shifting)** Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and

$$\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}.$$

14. Solve Prob. 6 by convolution. *Hint.* Show that $xe^{-x} = e^{-x} * e^{-x}$ $(x > 0)$.

15. Using Prob. 13, obtain formula 1 in Table III, Sec. 10.12, from formula 2 (with $c = 3b$).

16. Using the answer to Prob. 8, write down the Fourier integral representation of $f(x)$ and convert it to a Fourier cosine integral (see Sec. 10.9).

17. **(Shifting on the *w*-axis)** Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.

18. Using Prob. 17, obtain formula 7 in Table III, Sec. 10.12, from formula 1 in that table.

19. Using Prob. 17, obtain formula 8 in Table III, Sec. 10.12, from formula 2 in that table.

20. Verify formula 3 in Table III, Sec. 10.12, with $a = 1$. *Hint.* Use (15) in Sec. 10.9 and (3) in this section.

10.12 Tables of Transforms

For more extensive tables, see Ref.* [C4] *in Appendix 1.

Table I. Fourier Cosine Transforms

See (2) in Sec. 10.10.

	$f(x)$	$\hat{f}_c(w) = \mathcal{F}_c(f)$	
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin aw}{w}$	
2	x^{a-1} $(0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{w^a} \cos \frac{a\pi}{2}$	($\Gamma(a)$ see Appendix 3.1.)
3	e^{-ax} $(a > 0)$	$\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + w^2}\right)$	
4	$e^{-x^2/2}$	$e^{-w^2/2}$	
5	e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$	
6	$x^n e^{-ax}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + w^2)^{n+1}} \operatorname{Re}(a + iw)^{n+1}$	Re = Real part
7	$\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1 - w)}{1 - w} + \frac{\sin a(1 + w)}{1 + w}\right]$	
8	$\cos ax^2$ $(a > 0)$	$\frac{1}{\sqrt{2a}} \cos\left(\frac{w^2}{4a} - \frac{\pi}{4}\right)$	
9	$\sin ax^2$ $(a > 0)$	$\frac{1}{\sqrt{2a}} \cos\left(\frac{w^2}{4a} + \frac{\pi}{4}\right)$	
10	$\frac{\sin ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}}\, u(a - w)$	(See Sec. 6.3.)
11	$\frac{e^{-x} \sin x}{x}$	$\frac{1}{\sqrt{2\pi}} \arctan \frac{2}{w^2}$	
12	$J_0(ax)$ $(a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{u(a - w)}{\sqrt{a^2 - w^2}}$	(See Secs. 5.5, 6.3.)

Table II. Fourier Sine Transforms

See (5) in Sec. 10.10.

	$f(x)$	$\hat{f}_s(w) = \mathcal{F}_s(f)$
1	$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1 - \cos aw}{w}\right]$
2	$1/\sqrt{x}$	$1/\sqrt{w}$
3	$1/x^{3/2}$	$2\sqrt{w}$
4	x^{a-1} $(0 < a < 1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(a)}{w^a}\sin\frac{a\pi}{2}$ ($\Gamma(a)$ see Appendix 3.1.)
5	e^{-x}	$\sqrt{\frac{2}{\pi}}\left(\frac{w}{1 + w^2}\right)$
6	e^{-ax}/x $(a > 0)$	$\sqrt{\frac{2}{\pi}}\arctan\frac{w}{a}$
7	$x^n e^{-ax}$ $(a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{n!}{(a^2 + w^2)^{n+1}}\operatorname{Im}(a + iw)^{n+1}$ Im = Imaginary part
8	$xe^{-x^2/2}$	$we^{-w^2/2}$
9	xe^{-ax^2} $(a > 0)$	$\frac{w}{(2a)^{3/2}}e^{-w^2/4a}$
10	$\begin{cases} \sin x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1 - w)}{1 - w} - \frac{\sin a(1 + w)}{1 + w}\right]$
11	$\frac{\cos ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}}u(w - a)$ (See Sec. 6.3.)
12	$\arctan\frac{2a}{x}$ $(a > 0)$	$\sqrt{2\pi}\frac{\sinh aw}{w}e^{-aw}$

Table III. Fourier Transforms

See (6) in Sec. 10.11.

	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
1	$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin bw}{w}$
2	$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\|w\|}}{a}$
4	$\begin{cases} x & \text{if } 0 < x < b \\ 2x - a & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}\, w^2}$
5	$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$ $(a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
6	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
7	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin b(w - a)}{w - a}$
8	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{i}{\sqrt{2\pi}}\frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
9	e^{-ax^2} $(a > 0)$	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
10	$\frac{\sin ax}{x}$ $(a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\|w\| < a$; 0 if $\|w\| > a$

PROBLEM SET 5.8, page 248

3. $\lambda = [(2n+1)\pi/2]^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin(\frac{1}{2}(2n+1)\pi x)$
5. $\lambda = [(2n+1)\pi/2L]^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin((2n+1)\pi x/2L)$
7. $\lambda = n^2$, $n = 0, 1, \cdots$; $y_n(x) = \cos nx$
9. $\lambda = (n\pi/L)^2$, $n = 0, 1, \cdots$; $y_n(x) = \cos(n\pi x/L)$
11. $\lambda = ((2n+1)\pi/2)^2$, $n = 0, 1, \cdots$; $y_n(x) = \sin\left((2n+1)\frac{\pi}{2}\ln|x|\right)$
13. $\lambda = n^2\pi^2$, $n = 1, 2, \cdots$; $y_n(x) = x\sin(n\pi\ln|x|)$; Euler-Cauchy equation
15. $P_0/\sqrt{2}$, $\sqrt{\frac{3}{2}}P_1(x)$, $\sqrt{\frac{5}{2}}P_2(x)$
17. Set $x = ct + k$.

PROBLEM SET 5.9, page 255

1. $4P_1 + 2P_3$
3. $x^2 = \frac{1}{3}P_0 + \frac{2}{3}P_2$, $x^3 = \frac{3}{5}P_1 + \frac{2}{5}P_3$
5. $\frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{5}{16}P_2 + \cdots$
7. $\frac{1}{2}P_0 + \frac{5}{8}P_2 + \cdots$
15. Since the highest power in L_m is x^m, it suffices to show that $\int e^{-x}x^k L_n\,dx = 0$ for $k < n$:

$$\int_0^\infty e^{-x}x^k L_n(x)\,dx = \frac{1}{n!}\int_0^\infty x^k \frac{d^n}{dx^n}(x^n e^{-x})\,dx = -\frac{k}{n!}\int_0^\infty x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\,dx$$
$$= \cdots = (-1)^k\frac{k!}{n!}\int_0^\infty \frac{d^{n-k}}{dx^{n-k}}(x^n e^{-x})\,dx = 0.$$

21. $G_x = \sum a_n'(x)t^n = \sum He_n'(x)t^n/n! = tG = \sum He_{n-1}(x)t^n/(n-1)!$, etc.
23. Write $e^{-x^2/2} = v$, $v^{(n)} = d^n v/dx^n$, etc., integrate by parts, use the formula $He_n' = nHe_{n-1}$ (Prob. 21); then, for $n > m$,

$$\int_{-\infty}^{\infty} vHe_m He_n\,dx = (-1)^n\int_{-\infty}^{\infty} He_m v^{(n)}\,dx = (-1)^{n-1}\int_{-\infty}^{\infty} He_m' v^{(n-1)}\,dx$$
$$= (-1)^{n-1}m\int_{-\infty}^{\infty} He_{m-1}v^{(n-1)}\,dx = \cdots$$
$$= (-1)^{n-m}m!\int_{-\infty}^{\infty} He_0\, v^{(n-m)}\,dx = 0.$$

25. By (1) in Sec. 5.6, with $\nu = 1$,

$$a_m = \frac{2}{R^2 J_1^2(\alpha_{m0})}\int_0^R xJ_0\left(\frac{\alpha_{m0}}{R}x\right)dx = \frac{2}{\alpha_{m0}^2 J_1^2(\alpha_{m0})}\int_0^{\alpha_{m0}} wJ_0(w)\,dw$$
$$= \frac{2}{\alpha_{m0}J_1(\alpha_{m0})},\qquad f = 2\left(\frac{J_0(\lambda_{10}x)}{\alpha_{10}J_1(\alpha_{10})} + \frac{J_0(\lambda_{20}x)}{\alpha_{20}J_1(\alpha_{20})} + \cdots\right).$$

27. $a_m = \dfrac{2akJ_1(\alpha_{m0}a/R)}{\alpha_{m0}RJ_1^2(\alpha_{m0})}$
29. $a_m = \dfrac{4J_2(\alpha_{m0})}{\alpha_{m0}^2 J_1^2(\alpha_{m0})}$
31. $a_m = \dfrac{2R^2}{\alpha_{m0}J_1(\alpha_{m0})}\left[1 - \dfrac{2J_2(\alpha_{m0})}{\alpha_{m0}J_1(\alpha_{m0})}\right]$
35. $x^3 = 16\left[\dfrac{J_3(\alpha_{13}x/2)}{\alpha_{13}J_4(\alpha_{13})} + \dfrac{J_3(\alpha_{23}x/2)}{\alpha_{23}J_4(\alpha_{23})} + \cdots\right]$

CHAPTER 5 (REVIEW QUESTIONS AND PROBLEMS), page 258

17. $(1-x)^{-1}$, x^{-1}
19. e^x, $e^x\ln x$
21. $\sqrt{x}$, $1 + x$
23. $x + 1$, $(x+1)\ln x$
25. $x^2 + x^3$, $(x^2 + x^3)\ln x$
27. $[AJ_1(x) + BY_1(x)]/x$
29. $x[AJ_1(x) + BY_1(x)]$
31. $1/\sqrt{2}$, $\cos\pi x$, $\cos 2\pi x$, $\cdots$
33. $\sqrt{2/\pi}$, $(2/\sqrt{\pi})\cos 4mx$, $(2/\sqrt{\pi})\sin 4mx$
35. $\frac{1}{2}P_0(\frac{1}{2}x)$, $\frac{1}{2}\sqrt{3}P_1(\frac{1}{2}x)$, $\frac{1}{2}\sqrt{5}P_2(\frac{1}{2}x)$
37. $\lambda = n^2$, $y_n(x) = \sin(n\ln|x|)$
39. $\lambda = n^2\pi^2$, $y_n(x) = \cos(n\pi\ln|x|)$
41. $2(-P_0 + P_1 - P_2 + P_3)$
43. $-4P_0 + 10P_2 + 16P_4$
45. $-30P_0 + \frac{138}{5}P_1 - 6P_2 + \frac{2}{5}P_3$

PROBLEM SET 6.1, page 267

1. $3/s^2 + 4/s$
3. $2/s^3 + a/s^2 + b/s$
5. $(s\cos\theta - \omega\sin\theta)/(s^2 + \omega^2)$
7. $2n\pi[T(s^2 + (2n\pi/T)^2)]^{-1}$
9. $\cos^2\omega t = \frac{1}{2} + \frac{1}{2}\cos 2\omega t$. *Ans.* $1/2s + s/(2s^2 + 8\omega^2)$.
11. $1/2s + s/(2s^2 + 8)$
13. $e^b/(s - a)$
15. $\frac{1}{2}[s/(s^2 - 16) - 1/s]$
17. $k(1 - e^{-cs})/s$
19. $k/s - k(1 - e^{-cs})/cs^2$
27. $5e^{-3t}$
29. $\frac{1}{5}\sin 5t$
31. $t^3/6$
33. $1 - e^{-t}$
35. $3 - 3e^{-3t}$
37. $2 + e^{-2t}$

PROBLEM SET 6.2, page 274

1. $(s^2 + 2)/s(s^2 + 4)$
3. $(s - a)^{-2}$
5. $(s^2 - 8)/s(s^2 - 16)$
7. $(s^2 + 18)/s(s^2 + 36)$
15. $3 - 3e^{-t}$
17. $1 - \cos 2t$
19. $\sinh 2t - 2t$
21. $(e^{2t} - 1 - 2t - 2t^2)/8$
23. $t^2/2 - t/\pi + (e^{\pi t} - 1)/\pi^2$
25. $1 + t - \cos t - \sin t$
27. $2\cos 2t - 4\sin 2t$
29. $y = A\cos\omega t + (B/\omega)\sin\omega t$
31. $e^{-2t} - e^{-3t}$
33. $\cos 5t + \frac{1}{25}t$
35. $y = 2e^{kt}$

PROBLEM SET 6.3, page 281

1. $4.5/(s - 3.5)^2$
3. $1/(s^2 - 2s + 2)$
5. $(\omega\cos\theta + (s+1)\sin\theta)/[(s+1)^2 + \omega^2]$
7. $12/(s + \frac{1}{2})^4$
9. $[A(s + \alpha) + B\beta]/[(s + \alpha)^2 + \beta^2]$
11. $\pi t e^{-\pi t}$
13. $e^{2t}\cos t$
15. $e^{-3t}(\cos t - 3\sin t)$
17. $2e^{2t}\sinh 3t$
23. $k(e^{-as} - e^{-bs})/s$
25. $u(t) - u(t-1) + u(t-2) - + \cdots$,
$s^{-1}(1 - e^{-s} + e^{-2s} - + \cdots) = s^{-1}(1 + e^{-s})^{-1}$ (geometric series)

27. e^{-s}/s^2 **29.** $2e^{-s}/s^3$

31. $e^{1/2}e^{-s/2}/(s-1)$ **33.** $-se^{-\pi s}/(s^2+1)$

35. $t - tu(t-1) = t - (t-1)u(t-1) - u(t-1)$, $s^{-2} - s^{-2}e^{-s} - s^{-1}e^{-s}$

37. $(1 - e^{1-s})/(s-1)$ **39.** $s^{-2} - e^{-as}(s^{-2} + as^{-1})$

41. $-2s(e^{-s} + e^{-2s})/(s^2 + \pi^2)$ **43.** $t - 3$ if $t > 3$; 0 if $t < 3$

45. -3 if $1 < t < 4$; 0 otherwise **47.** $\cos 2t$ if $t > \pi$; 0 otherwise

49. $(t-1)^3/6$ if $t > 1$; 0 otherwise **51.** $e^{-t}\sin t$

53. $3e^{t/2}\cos 3t$ **55.** $e^{-2t}\sin 3t + 9\cos 2t + 8\sin 2t$

57. $e^{-2t} + e^{-3t} + e^{-4t} + e^{-5t}$

59. $t - \sin t$ if $0 < t < 1$; $\cos(t-1) + \sin(t-1) - \sin t$ if $t > 1$

61. $\sin 3t + \sin t$ if $0 < t < \pi$; $\frac{4}{3}\sin 3t$ if $t > \pi$

63. $\sin 2t + \sin t$ if $0 < t < \pi$; $-\sin t$ if $t > \pi$

65. $e^t - \sin t$ if $0 < t < 2\pi$; $e^t - \frac{1}{2}\sin 2t$ if $t > 2\pi$

67. $i(t) = V_0(1 - e^{-Rt/L})/R$

69. $sI + I/s = (1 - e^{-s})/s^2$. *Ans.* $1 - \cos t$ if $0 < t < 1$, $\cos(t-1) - \cos t$ if $t > 1$.

PROBLEM SET 6.4, page 288

1. $i = 0\ (t < 1)$, $i = e^{-0.1(t-1)}\ (1 < t < 2)$, $i = e^{-0.1(t-1)} - e^{-0.1(t-2)}\ (t > 2)$

3. $i = 0\ (t < 2)$, $i = (10e^{-(t-2)} - e^{-0.1(t-2)})/900e^2\ (t > 2)$

5. $i = 20(1 - e^{-0.1t})\ (0 < t < 1)$, $i = 20(e^{0.1} - 1)e^{-0.1t}\ (t > 1)$

7. $y = 0\ (t < 1)$, $y = \frac{1}{3}\sin(3t - 3)\ (t > 1)$

9. $y = \sin t\ (0 < t < \pi)$, $y = 0\ (\pi < t < 2\pi)$, $y = -\sin t\ (t > 2\pi)$

11. $y = e^{-t}\cos t\ (0 < t < 2\pi)$, $y = e^{-t}(\cos t + e^2 \sin t)\ (t > 2\pi)$

13. $y = 3e^{-2t}\sin t\ (0 < t < 1)$, $y = e^{-2t}[3\sin t + e^2\sin(t-1)]\ (t > 1)$

15. $y = 2e^{-3t} + e^{-t}\ (0 < t < \frac{1}{2})$, $y = 2e^{-3t} + e^{-t} + \frac{1}{4}[e^{-3(t-1/2)} - e^{t-1/2}]\ (t > \frac{1}{2})$

17. $y = e^{-t}\cos 2t + e^t\ (0 < t < 1)$,
$y = e^{-t}\cos 2t + e^t + e^{-(t-1)}\sin 2(t-1)\ (t > 1)$

19. $y = 5t - 2\ (0 < t < \pi)$, $y = 5t - 2 - \frac{1}{2}e^{-(t-\pi)}\sin 2t\ (t > \pi)$

PROBLEM SET 6.5, page 293

1. $\dfrac{2s^2 - 8}{(s^2+4)^2}$ **3.** $\dfrac{2}{(s-1)^3}$ **5.** $\dfrac{6s}{(s^2-9)^2}$

7. $\dfrac{2\omega s}{(s^2+\omega^2)^2}$ **9.** $\dfrac{2s(s^2-3)}{(s^2+1)^3}$ **11.** $\dfrac{s+2}{(s^2+4s+5)^2}$

13. $4e^{-t}t$ **15.** $t\sinh 2t$ **17.** $\frac{1}{2}t\sin t$

19. $(e^t - 1)/t$ **21.** $(\sin \omega t)/t$

PROBLEM SET 6.6, page 297

1. t **3.** te^t **5.** $t^5/30$ **7.** $\frac{1}{2}t\sin \omega t$

9. $\frac{2}{3}\sin t - \frac{1}{3}\sin 2t$ **11.** $0\ (t < \pi)$, $-\sin t\ (t > \pi)$

13. $e^t * e^t = te^t$ **15.** $(e^{3t} - 1)/9 - t/3$

17. $(1 - \cos \omega t)/\omega^2$ **19.** $(t \sin \omega t)/2\omega$ **21.** $t\cos \omega t$

23. $e^{-t} - e^{-2t}$

25. Set $t - \tau = \sigma$. Then $\int_0^t f(\tau)g(t-\tau)\,d\tau = \int_t^0 g(\sigma)f(t-\sigma)(-d\sigma)$.

29. Let $t > k$. Then $(f_k * f)(t) = \int_0^k \frac{1}{k} f(t-\tau)\,d\tau = f(t - \tilde{t})$ for some $\tilde{t}$ between 0 and k. Now let $k \to 0$. Then $\tilde{t} \to 0$ and $f_k(t - \tilde{t}) \to \delta(t)$, so that the formula follows.

31. $\frac{3}{8}\sin t - \frac{1}{8}\sin 3t$ **33.** $t - \sin t$

35. $0.2e^{-t} + \cos 5t - 2\sin 5t$

39. 0 if $0 < t < 1$; $\frac{1}{4} - \frac{1}{4}\cos(2t - 2)$ if $t > 1$

41. $\sin t + \sin 3t$ if $0 < t < \pi$; $\frac{4}{3}\sin 3t$ if $t > \pi$

43. $\sin t + \sin 2t$ if $0 < t < \pi$; $-\sin t$ if $t > \pi$

45. e^t **47.** $\cos t$ **49.** $\sinh t$

PROBLEM SET 6.7, page 308

1. $(e^{4t} - e^t)/3$ **3.** $2e^{-4t} + e^{2t}$

5. $e^{-t}(\cos 3t + 4\sin 3t)$ **7.** $2 + e^t - 2e^{2t}$

9. $e^{2t}(2t - 4)$ **11.** $e^{-t}(1 - t - t^2)$

13. $\cosh 3t - 3\cosh t$ **15.** $e^{-t}(\cos 2t - 2t\sin 2t)$

17. $(1 + t^2 - 2t^3)e^{-2t}$ **19.** $2e^{2t} + te^t$

27. $y_1 = \cos t$, $y_2 = \sin t$ **29.** $y_1 = e^{-t}\cos t$, $y_2 = -e^{-t}\sin t$

31. $y_1 = -6e^{4t} + 2$, $y_2 = -3e^{4t} - 1$ **33.** $y_1 = 4e^t - e^{-2t}$, $y_2 = e^t - e^{-2t}$

35. $y_1 = 3e^{2t} + e^{-5t}$, $y_2 = 4e^{2t} - e^{-5t}$ **37.** $y_1 = e^t + e^{2t}$, $y_2 = e^{2t}$

39. $y_1 = e^t$, $y_2 = e^{-t}$, $y_3 = e^t - e^{-t}$

PROBLEM SET 6.8, page 313

1. $\dfrac{\pi s - 1 + (\pi s + 1)e^{-2\pi s}}{s^2(1 - e^{-2\pi s})} = \dfrac{\pi}{s}\coth \pi s - \dfrac{1}{s^2}$

3. $\dfrac{(4\pi^2 s^2 - 2)e^{2\pi s} + 4\pi s + 2}{s^3(e^{2\pi s} - 1)}$ **5.** $\dfrac{e^{2(1-s)\pi} - 1}{(1-s)(1 - e^{-2\pi s})}$

7. $\left[\dfrac{1}{s^2}(1 - e^{-\pi s}) - \dfrac{\pi}{s}e^{-\pi s}\right]\Big/(1 - e^{-2\pi s})$

9. $\left[\dfrac{\pi}{s}e^{-\pi s}(e^{-\pi s} - 1) + \dfrac{1}{s^2}(e^{-\pi s} - 1)^2\right]\Big/(1 - e^{-2\pi s})$

15. $\dfrac{\omega}{s^2 + \omega^2}\coth\dfrac{\pi s}{2\omega}$

21. $i(t) = (V_0/\omega^* L)e^{-\alpha t}\sin \omega^* t$ if $0 < t < a$, $\alpha = R/2L$, $\omega^{*2} = 1/LC - \alpha^2$,
$i(t) = (V_0/\omega^* L)[e^{-\alpha t}\sin \omega^* t - e^{-\alpha(t-a)}\sin\{\omega^*(t-a)\}]$ if $t > a$

25. $i(t) = \dfrac{1}{R}\left(\dfrac{e^{-kt}}{1 - e^{-k}} + t - \dfrac{L}{R}\right)$ if $0 < t < 1$, and $i(t+1) = i(t)$, $k = \dfrac{R}{L}$.

1. $3a^2$ 3. 24π 5. 1 7. $2 \cdot 5^7/7$ 9. 10/3
11. $-\frac{1}{6} + e^{-1} + \frac{1}{2}e^2 - \frac{1}{3}e^3$ 13. $-1/3$ 15. $9a^4\pi/16$
17. $a^2(3\pi/8 - 1)$ 19. $4e^2 - 4$ 21. 80π
23. 2 25. $2e - 15/4$ 27. $4e^4 - 4$ 29. 16π

PROBLEM SET 9.5, page 533

1. Straight lines, **k**
3. $x^2/16 + y^2/4 = 1$, straight lines, ellipses, $2\cos u\,\mathbf{i} + 4\sin u\,\mathbf{j}$
5. $z = c\sqrt{x^2 + y^2}$, circles, straight lines, $-cu\cos v\,\mathbf{i} - cu\sin v\,\mathbf{j} + u\mathbf{k}$
7. $z = x^2 + y^2$, circles, parabolas, $-2u^2\cos v\,\mathbf{i} - 2u^2\sin v\,\mathbf{j} + u\mathbf{k}$
9. $z = x^2/a^2 + y^2/b^2$, ellipses, parabolas, $-2bu^2\cos v\,\mathbf{i} - 2au^2\sin v\,\mathbf{j} + abu\mathbf{k}$
11. $x^2/a^2 + y^2/b^2 - z^2/c^2 + 1 = 0$, ellipses, hyperbolas, $-bc\sinh^2 u\cos v\,\mathbf{i} - ac\sinh^2 u\sin v\,\mathbf{j} + ab\cosh u\sinh u\,\mathbf{k}$
13. $u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$, $-\mathbf{j} + \mathbf{k}$
15. $u\mathbf{i} + v\mathbf{j} + (1 - u - v)\mathbf{k}$, $\mathbf{i} + \mathbf{j} + \mathbf{k}$
17. $\cosh u\,\mathbf{i} + \sinh u\,\mathbf{j} + v\mathbf{k}$, $\cosh u\,\mathbf{i} - \sinh u\,\mathbf{j}$
19. $u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + 16u^2\mathbf{k}$, $-32u^2\cos v\,\mathbf{i} - 32u^2\sin v\,\mathbf{j} + u\mathbf{k}$
21. $u\mathbf{i} + v\mathbf{j} + 16(u^2 + v^2)\mathbf{k}$, $-32u\mathbf{i} - 32v\mathbf{j} + \mathbf{k}$
25. $\mathbf{r}_u$ is tangent to the curves $v = const$, and $\mathbf{r}_v$ is tangent to $u = const.$
27. $(1/\sqrt{38})(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$
29. $(1/3)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
31. $(1/a)(x\mathbf{i} + y\mathbf{j})$
33. $(x^2 + y^2 + 1)^{-1/2}(-y\mathbf{i} - x\mathbf{j} + \mathbf{k})$
37. $z^* = 2\sqrt{2} + y^*$
39. $3x^* + 4y^* = 25$

PROBLEM SET 9.6, page 542

1. -16 3. $2e^3 - 3e^2 + 1$ 5. 7π 7. -1250π
9. $17h/4$ 11. $-59/180$ 13. $-53/2$
15. $2\cosh^3 2 - 2 \approx 104.5$ 17. $3\sqrt{3}$ 19. $\pi a^3h^3/6$
21. $17^{5/2}\pi/6$ 23. $10^{3/2} - 1$ 25. $3^{5/2} - 2^{7/2} + 1$
29. $2\pi h$ 31. $\pi h^4/\sqrt{2}$ 33. $h\pi + 2h^3\pi/3$
35. Use $d\mathbf{r} = \mathbf{r}_u\,du + \mathbf{r}_v\,dv$.

PROBLEM SET 9.7, page 549

1. 8/27 3. 96π 5. 1/120 7. $8\pi^2$ 9. 19/180
11. 2/3 13. $\pi ha^4/2$ 15. $8\pi a^5/15$ 17. 16 19. $6\pi a^2h$
21. 2π 23. 1/6 25. $40\pi/3$ 27. $384\pi/5$ 29. $-\pi/8$

PROBLEM SET 9.8, page 555

3. Put $f = g$ in (10). 5. Use (11). 9. $r = a$, $\cos\phi = 1$

PROBLEM SET 9.9, page 561

1. ± 3 3. ± 6 5. $\pm(e^2 - 1)$
7. $\pm 2(1 - e^2)$ 11. 24π 13. 2π
15. $-1/3$ 17. $-2/3$ 19. $-18\pi\sqrt{2}$

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17. 2/3 19. $e^2/2$ 21. $\pm 25\pi$ 23. 0 25. $e^4 + 15$
27. $\pm 12\pi$ 29. 0 31. $8e^{-24}$ 33. 0
35. $\pm(1 - \cosh 1)$ 37. $(e^2 - 1)^2/2$ 39. 1/8
41. 189/20 43. 4/5, 8/15 45. $8a/5\pi$, $8a/5\pi$
47. $3\cos u\,\mathbf{i} + 3\sin u\,\mathbf{j} + v\mathbf{k}$, $3\cos u\,\mathbf{i} + 3\sin u\,\mathbf{j}$
49. $u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\mathbf{k}$, $-u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\mathbf{k}$
51. $2u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u^2\mathbf{k}$, $-2u^2\cos v\,\mathbf{i} - 4u^2\sin v\,\mathbf{j} + 2u\mathbf{k}$
53. 5/216 55. 4/9 57. $-4/3 + 8\pi/3$ 59. 40

PROBLEM SET 10.1, page 568

1. $2\pi, 2\pi, \pi, \pi, 2, 2, 1, 1$
17. 0 (n even), $2/n$ (n odd)
19. 0
21. $n[(-1)^n e^{-\pi} - 1]/(1 + n^2)$
23. $[(-1)^n e^{\pi} - 1]/(1 + n^2)$
25. 0 $(n = 0)$, π/n $(n = 1, 3, \cdots)$, $-\pi/n$ $(n = 2, 4, \cdots)$

PROBLEM SET 10.2, page 576

1. $\frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$
3. $\frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$
5. $\frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$
7. $2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right)$
9. $\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$
11. $2\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin x - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2x + \left(\frac{\pi^2}{3} - \frac{6}{3^3}\right)\sin 3x - + \cdots\right]$
13. $\frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots$
15. $\frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right)$

PROBLEM SET 10.3, page 580

1. $\frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right)$

3. $1 + \frac{4}{\pi}\left(\sin \frac{\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} + \frac{1}{5}\sin \frac{5\pi x}{2} + \cdots\right)$

5. $\frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$

7. $\frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) + \frac{1}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

9. $-\frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) + \frac{2}{\pi}\left(2\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

11. $1 - \frac{12}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - \frac{1}{16}\cos 4\pi x + - \cdots\right)$

13. $4\left(\frac{1}{2} - \frac{1}{1\cdot 3}\cos 2\pi x - \frac{1}{3\cdot 5}\cos 4\pi x - \frac{1}{5\cdot 7}\cos 6\pi x - \cdots\right)$

17. Write τ for x in Example 1 and set $\tau = x - 1$.

PROBLEM SET 10.4, page 584

1. Even: $|x^3|$, $x^2 \cos nx$, $\cosh x$. Odd: $x \cos nx$, $\sinh x$, $x|x|$.
3. Odd 5. Odd 7. Even 9. Neither 11. Neither
13. $1/(1 - x^2) + x/(1 - x^2)$ 15. $\cosh kx + \sinh kx$

25. $\frac{k}{2} + \frac{2k}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - + \cdots\right)$

27. $\frac{4}{\pi}\left(\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - + \cdots\right)$

29. $\frac{\pi^2}{6} - \frac{4}{\pi}\cos x - \frac{2}{2^2}\cos 2x + \frac{4}{3^3\pi}\cos 3x + \frac{2}{4^2}\cos 4x - \frac{4}{5^3\pi}\cos 5x + \cdots$

31. $\frac{\pi^2}{12} - \cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - + \cdots$

PROBLEM SET 10.5, page 588

1. $\frac{4k}{\pi}\left(\sin \frac{\pi x}{L} + \frac{1}{3}\sin \frac{3\pi x}{L} + \frac{1}{5}\sin \frac{5\pi x}{L} + \cdots\right)$

3. $\frac{2L^2}{\pi}\left[\left(1 - \frac{4}{\pi^2}\right)\sin \frac{\pi x}{L} - \frac{1}{2}\sin \frac{2\pi x}{L} + \left(\frac{1}{3} - \frac{4}{3^3\pi^2}\right)\sin \frac{3\pi x}{L} - \frac{1}{4}\sin \frac{4\pi x}{L} + \cdots\right]$

5. $\frac{2L}{\pi}\left(\sin \frac{\pi x}{L} + \frac{1}{2}\sin \frac{2\pi x}{L} + \frac{1}{3}\sin \frac{3\pi x}{L} + \cdots\right)$

7. $\left(1 + \frac{2}{\pi}\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3} - \frac{2}{9\pi}\right)\sin 3x - \frac{1}{4}\sin 4x + \cdots$

9. $\frac{4L}{\pi^2}\left(\frac{1}{1^2}\sin \frac{\pi x}{L} - \frac{1}{3^2}\sin \frac{3\pi x}{L} + \frac{1}{5^2}\sin \frac{5\pi x}{L} - + \cdots\right)$

11. $\frac{L}{2} - \frac{4L}{\pi^2}\left(\cos \frac{\pi x}{L} + \frac{1}{9}\cos \frac{3\pi x}{L} + \frac{1}{25}\cos \frac{5\pi x}{L} + \cdots\right)$

13. $\frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos \frac{\pi x}{L} - \frac{1}{4}\cos \frac{2\pi x}{L} + \frac{1}{9}\cos \frac{3\pi x}{L} - \frac{1}{16}\cos \frac{4\pi x}{L} + - \cdots\right)$

15. $\frac{1}{2} - \frac{2}{\pi}\left(\cos \frac{\pi x}{L} - \frac{1}{3}\cos \frac{3\pi x}{L} + \frac{1}{5}\cos \frac{5\pi x}{L} - + \cdots\right)$

17. $a_0 = \frac{1}{L}(e^L - 1)$, $a_n = \frac{2L}{L^2 + n^2\pi^2}[(-1)^n e^L - 1]$

19. $\frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3}\cos \frac{2\pi x}{L} + \frac{1}{3\cdot 5}\cos \frac{4\pi x}{L} + \frac{1}{5\cdot 7}\cos \frac{6\pi x}{L} + \cdots\right)$

PROBLEM SET 10.6, page 591

1. Use (7).

3. $-\frac{2i}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{2n+1}e^{(2n+1)ix}$

5. $i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n}{n}e^{inx}$

7. $\pi + i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}e^{inx}$

9. $\frac{\pi^2}{3} + 2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^n}{n^2}e^{inx}$

PROBLEM SET 10.7, page 594

3. $y = C_1 \cos \omega t + C_2 \sin \omega t + A(\omega)\sin t$, $A(\omega) = 1/(\omega^2 - 1)$, $A(0.5) = -1.33$, $A(0.7) = -1.96$, $A(0.9) = -5.3$, $A(1.1) = 4.8$, $A(1.5) = 0.8$, $A(2) = 0.33$, $A(10) = 0.01$

5. $y = C_1 \cos \omega t + C_2 \sin \omega t + \sum_{n=1}^{N}\frac{a_n}{\omega^2 - n^2}\cos nt$

7. $y = C_1 \cos \omega t + C_2 \sin \omega t + \frac{\pi}{2\omega^2} + \frac{4}{\pi}\left(\frac{1}{\omega^2 - 1}\cos t + \frac{1}{9(\omega^2 - 9)}\cos 3t + \cdots\right)$

9. $y = \frac{1 - n^2}{D}a_n \cos nt + \frac{nc}{D}a_n \sin nt$, $D = (1 - n^2)^2 + n^2c^2$

11. $y = -\frac{3c}{64 + 9c^2}\cos 3t - \frac{8}{64 + 9c^2}\sin 3t$

13. $y = \sum_{n=1}^{\infty}\left[\frac{(-1)^n c}{n^2 D_n}\cos nt - \frac{(-1)^n(1 - n^2)}{n^3 D_n}\sin nt\right]$, $D_n = (1 - n^2)^2 + n^2c^2$

15. $I = \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$, $A_n = (-1)^{n+1}\frac{240(10 - n^2)}{n^2 D_n}$, $B_n = \frac{(-1)^{n+1}2400}{nD_n}$, $D_n = (10 - n^2)^2 + 100n^2$

PROBLEM SET 10.8, page 598

1. $F = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{N}\sin Nx\right]$ (N odd)

5. $F = 2\left(\sin x - \frac{1}{2}\sin 2x + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx\right)$,
$E^* \approx 8, 5, 3.6, 2.8, 2.3$

7. $F = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \cdots + \frac{(-1)^{N+1}}{N^2}\cos Nx\right)$,
$E^* \approx 4.14, 1.00, 0.38, 0.18, 0.10$

9. $F = \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots$,
$E^* = \frac{\pi^3}{12} - \pi\left[\frac{4}{\pi^2} + \frac{1}{4} + \frac{4}{81\pi^2} - \frac{1}{16} + \frac{4}{625\pi^2} + \cdots\right]$; 1.311, 0.525, 0.509, 0.313, 0.311

15. Use the Fourier series $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$.

PROBLEM SET 10.9, page 605

9. $\frac{2}{\pi}\int_0^\infty \left[\left(1 - \frac{2}{w^2}\right)\sin w + \frac{2}{w}\cos w\right]\frac{\cos wx}{w}\,dw$

11. $\frac{2}{\pi}\int_0^\infty \left[\frac{a \sin aw}{w} + \frac{\cos aw - 1}{w^2}\right]\cos xw\,dw$

13. $A = \frac{2}{\pi}\int_0^\infty \frac{\cos wv}{1 + v^2}\,dv = e^{-w}\ (w > 0),\quad f(x) = \int_0^\infty e^{-w}\cos wx\,dw$

15. $f(ax) = \int_0^\infty A(w)\cos axw\,dw = \int_0^\infty A\left(\frac{p}{a}\right)\cos xp\,\frac{dp}{a}$, where $wa = p$.
If we write again w instead of p, the result follows.

17. Differentiating (10) we have $\frac{d^2A}{dw^2} = -\frac{2}{\pi}\int_0^\infty f^*(v)\cos wv\,dv$, $f^*(v) = v^2 f(v)$, and the result follows.

PROBLEM SET 10.10, page 610

1. $\sqrt{2/\pi}\,(\sin 2w - 2\sin w)/w$
3. $\sqrt{2/\pi}\,(aw \sin aw + \cos aw - 1)/w^2$
7. $e^{-w}\sqrt{\pi/2}$
9. $\sqrt{2/\pi}\,[(2 - w^2)\cos w + 2w \sin w - 2]/w^3$
11. $\sqrt{\pi/2}$ if $0 < w < \pi$, 0 if $w > \pi$
13. $\sqrt{\pi/2}\cos w$ if $|w| < \pi/2$, 0 if $|w| > \pi/2$
17. $\sqrt{\pi/2}\,e^{-w}\cos w$
19. No

PROBLEM SET 10.11, page 618

1. $1/(1 + iw)\sqrt{2\pi}$
3. $\sqrt{2/\pi}\,(2 - w)^{-1}\sin(2 - w)$
5. $[-1 + (1 + iaw)e^{-iaw}]/w^2\sqrt{2\pi}$
7. $i\sqrt{2/\pi}\,(\cos w - 1)/w$

CHAPTER 10 (REVIEW QUESTIONS AND PROBLEMS), page 622

17. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

19. $\frac{2}{\pi}\left(\sin x - \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x - \frac{2}{6}\sin 6x + \cdots\right)$

21. $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

23. $-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x - + \cdots$

25. $\frac{8}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

27. $\frac{\pi^2}{12} - \frac{2}{\pi}\cos x - \frac{1}{2^2}\cos 2x + \frac{2}{3^3\pi}\cos 3x + \frac{1}{4^2}\cos 4x - \cdots$

29. $\frac{8}{\pi}\left(\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots\right)$

31. $-\frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}5\pi x + \cdots\right)$

33. $\frac{4}{\pi}\left(\sin \frac{\pi}{2}x - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin \frac{3\pi}{2}x - + \cdots\right)$

35. $\frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) - \frac{1}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

37. $-\frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) + \frac{2}{\pi}\left(2\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

39. $-\frac{1}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right) + \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

41. $\pi/4$
43. $\pi^3/32$
47. 5.168, 0.075, 0.075, 0.012, 0.012, 0.004

49. $y = C_1 \cos \omega t + C_2 \sin \omega t + \frac{\pi^2}{12\omega^2} - \frac{1}{\omega^2 - 1}\cos t + \frac{1}{4(\omega^2 - 4)}\cos 2t - + \cdots$

PROBLEM SET 11.1, page 628

25. $u = f(x)$
27. $u_x = f(y)$, $u = xf(y) + g(y)$
29. $u = c(y)e^{x^2 y}$
31. $u = v(x) + w(y)$
33. $u = c = const$
35. $u = cx + g(y)$

PROBLEM SET 11.3, page 637

1. $u = 0.02 \cos t \sin x$
3. $u = k(\cos t \sin x - \cos 2t \sin 2x)$

5. $u = \frac{4}{5\pi}\left(\frac{1}{4}\cos 2t \sin 2x - \frac{1}{36}\cos 6t \sin 6x + \frac{1}{100}\cos 10t \sin 10x - + \cdots\right)$

7. $u = \frac{8k}{\pi}\left(\cos t \sin x + \frac{1}{3^3}\cos 3t \sin 3x + \frac{1}{5^3}\cos 5t \sin 5x + \cdots\right)$