We know that if an *ordinary* differential equation is linear and homogeneous, then from known solutions we can obtain further solutions by superposition. For a homogeneous linear *partial* differential equation the situation is quite similar. In fact, the following theorem holds.

Theorem 1 Fundamental Theorem (Superposition or linearity principle)

If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region R, then

$$u = c_1u_1 + c_2u_2,$$

where c_1 and c_2 are any constants, is also a solution of that equation in R.

The proof of this important theorem is simple and quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–23 proceeds as for ordinary differential equations. Problems 24–35 concern partial differential equations that can be solved like ordinary ones; to help the student with them, we consider two typical examples.

EXAMPLE 2 Find a solution $u(x, y)$ of the partial differential equation $u_{xx} - u = 0$.

Solution. Since no y-derivatives occur, we can solve this like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^x + Be^{-x}$ with constant A and B. Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)e^x + B(y)e^{-x}$$

with arbitrary functions A and B, so that we have a great variety of solutions. Check the result by differentiation. ∎

EXAMPLE 3 Solve the partial differential equation $u_{xy} = -u_x$.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln p = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x,

$$u(x, y) = f(x)e^{-y} + g(y) \qquad \text{where} \qquad f(x) = \int c(x)\,dx;$$

here, $f(x)$ and $g(y)$ are arbitrary. ∎

Problem Set 11.1

1. Prove Fundamental Theorem 1 for second-order differential equations in two and three independent variables.
2. Verify that the functions (6) are solutions of (3).

Verify that the following functions are solutions of Laplace's equation.

3. $u = 2xy$
4. $u = x^3 - 3xy^2$
5. $u = x^4 - 6x^2y^2 + y^4$
6. $u = e^x \sin y$
7. $u = \sin x \sinh y$
8. $u = \arctan(y/x)$

Verify that the following functions are solutions of the wave equation (1) for a suitable value of c.

9. $u = x^2 + 4t^2$
10. $u = x^3 + 3xt^2$
11. $u = \sin 2ct \sin 2x$
12. $u = \cos 4t \sin x$
13. $u = \cos ct \sin x$
14. $u = \sin \omega ct \sin \omega x$

Verify that the following functions are solutions of the heat equation (2) for a suitable value of c.

15. $u = e^{-t} \cos x$
16. $u = e^{-2t} \cos x$
17. $u = e^{-t} \sin 3x$
18. $u = e^{-4t} \cos \omega x$
19. $u = e^{-16t} \cos 2x$
20. $u = e^{-\omega^2c^2t} \sin \omega x$

21. Show that $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of Laplace's equation (5).
22. Verify that $u(x, y) = a \ln (x^2 + y^2) + b$ satisfies Laplace's equation (3) and determine a and b so that u satisfies the boundary conditions $u = 0$ on the circle $x^2 + y^2 = 1$ and $u = 5$ on the circle $x^2 + y^2 = 9$.
23. Show that $u(x, t) = v(x + ct) + w(x - ct)$ is a solution of the wave equation (1); here, v and w are any twice differentiable functions.

Partial differential equations solvable as ordinary differential equations

If an equation involves derivatives with respect to one variable only, we can solve it like an ordinary differential equation, treating the other variable (or variables) as parameters. Find solutions $u(x, y)$ of

24. $u_x = 0$
25. $u_y = 0$
26. $u_{xx} + 4u = 0$
27. $u_{xx} = 0$
28. $u_y + 2yu = 0$
29. $u_x = 2xyu$

Setting $u_x = p$, solve

30. $u_{xy} = u_x$
31. $u_{xy} = 0$
32. $u_{xyy} + u_x = 0$

Solve the following systems of partial differential equations.

33. $u_x = 0, \quad u_y = 0$
34. $u_{xx} = 0, \quad u_{yy} = 0$
35. $u_{xx} = 0, \quad u_{xy} = 0$

11.2 Modeling: Vibrating String, Wave Equation

As a first important partial differential equation, let us derive the equation governing small transverse vibrations of an elastic string, such as a violin string. We stretch the string to length L and fix it at the ends. We then distort it and at some instant, say, $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 251.

When deriving a differential equation corresponding to a given physical problem, we usually have to make simplifying assumptions to ensure that the resulting equation does not become too complicated. We know this important fact from our study of ordinary differential equations, and for partial differential equations the situation is similar.

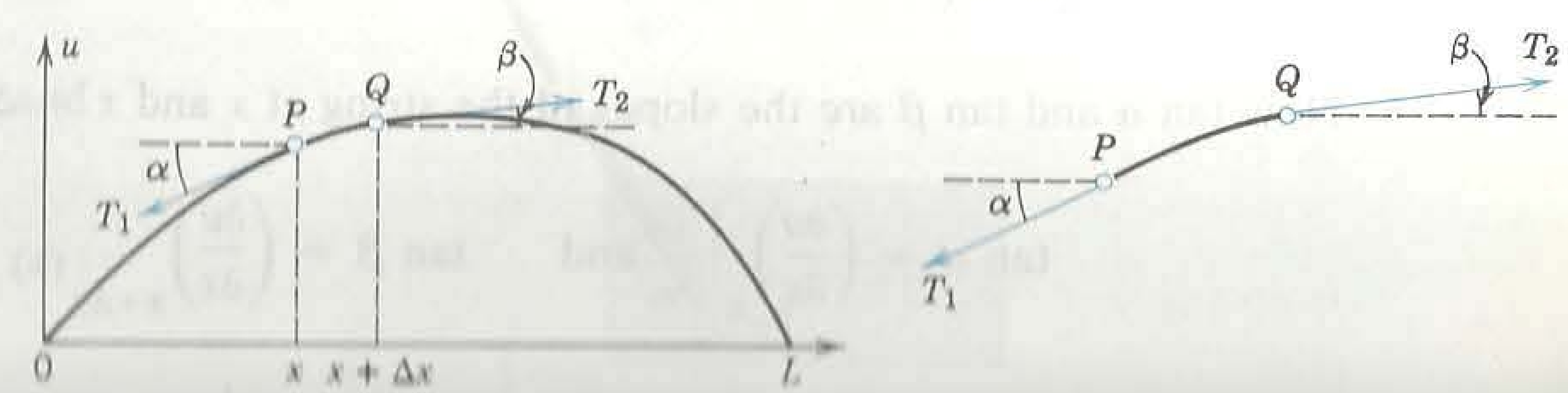


Fig. 251. Deflected string at fixed time t

identically zero. Then the B_n^* are zero, and (12) reduces to

(16) $$u(x, t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}.$$

It is possible to *sum this series,* that is, to write the result in a closed or finite form. For this purpose we use the formula [see (11), Appendix 3.1]

$$\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin \left\{\frac{n\pi}{L}(x - ct)\right\} + \sin \left\{\frac{n\pi}{L}(x + ct)\right\}\right].$$

Consequently, we may write (16) in the form

$$u(x, t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin \left\{\frac{n\pi}{L}(x - ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin \left\{\frac{n\pi}{L}(x + ct)\right\}.$$

These two series are those obtained by substituting $x - ct$ and $x + ct$, respectively, for the variable x in the Fourier sine series (13) for $f(x)$. Thus

(17) $$u(x, t) = \tfrac{1}{2}[f^*(x - ct) + f^*(x + ct)]$$

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 254). Since the initial deflection $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and zero at the endpoints, it follows from (17) that $u(x, t)$ is a continuous function of both variables x and t for all values of the variables. By differentiating (17) we see that $u(x, t)$ is a solution of (1), provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$, which are zero. Under these conditions $u(x, t)$ is established as a solution of (1), satisfying (2)–(4). ■

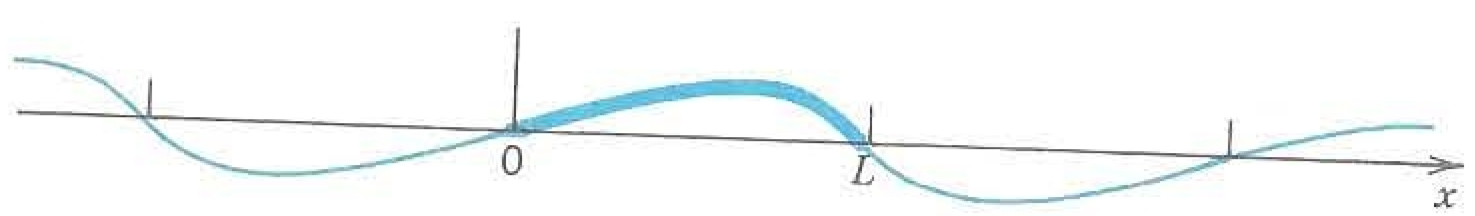


Fig. 254. Odd periodic extension of $f(x)$

If $f'(x)$ and $f''(x)$ are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each t there will be finitely many values of x at which the second derivatives of u appearing in (1) do not exist. Except at these points the wave equation will still be satisfied, and we may then regard $u(x, t)$ as a **"generalized solution,"** as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

Representation (17) has an interesting physical interpretation, as follows. The graph of $f^*(x - ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right (Fig. 255). This means that $f^*(x - ct)$ $(c > 0)$

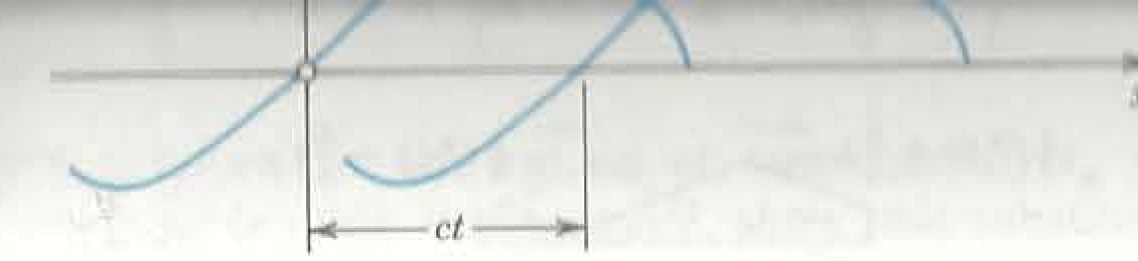


Fig. 255. Interpretation of (17)

represents a wave that is traveling to the right as t increases. Similarly, $f^*(x + ct)$ represents a wave that is traveling to the left, and $u(x, t)$ is the superposition of these two waves.

EXAMPLE 1 **Vibrating string if the initial deflection is triangular**

Find the solution of the wave equation (1) corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \dfrac{2k}{L} x & \text{if } 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{if } \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 256 on p. 638 shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 1 in Sec. 10.5 we see that the B_n are given by (5), Sec. 10.5. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2} \sin \frac{\pi}{L} x \cos \frac{\pi c}{L} t - \frac{1}{3^2} \sin \frac{3\pi}{L} x \cos \frac{3\pi c}{L} t + - \cdots\right].$$

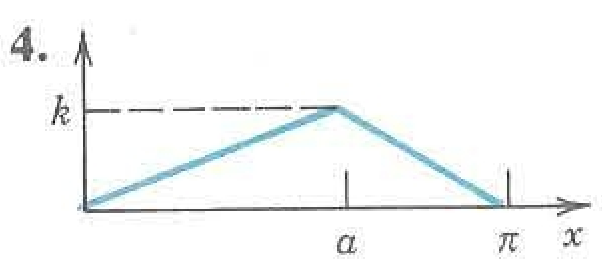
For plotting the graph of the solution we may use $u(x, 0) = f(x)$ and the above interpretation of the two functions in the representation (17). This leads to the graph shown in Fig. 256. ■

Problem Set 11.3

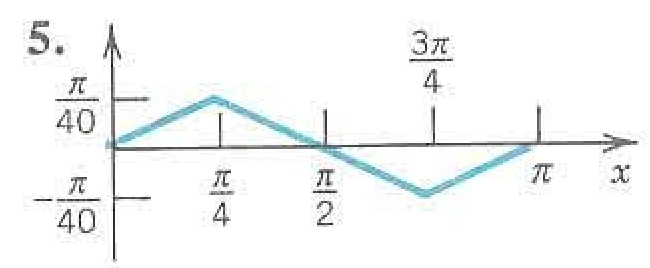
Find the deflection $u(x, t)$ of the vibrating string (length $L = \pi$, ends fixed, and $c^2 = T/\rho = 1$) corresponding to zero initial velocity and initial deflection:

1. $0.02 \sin x$ **2.** $k \sin 3x$ **3.** $k(\sin x - \sin 2x)$

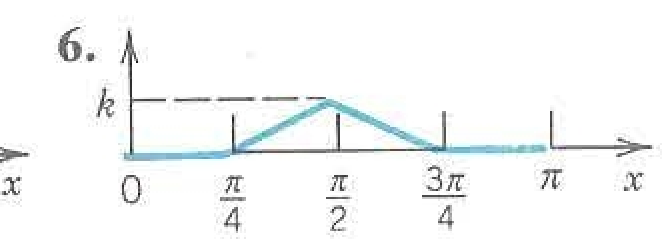
4.



5.



6.



7. $k(\pi x - x^2)$ **8.** $k(\pi^2 x - x^3)$ **9.** $k[(\frac{1}{2}\pi)^4 - (x - \frac{1}{2}\pi)^4]$

Find the deflection $u(x, t)$ of the vibrating string (length $L = \pi$, ends fixed, $c^2 = 1$) if the initial deflection $f(x)$ and the initial velocity $g(x)$ are

10. $f = 0$, $g(x) = 0.1 \sin 2x$ **11.** $f(x) = 0.1 \sin x$, $g(x) = -0.2 \sin x$

12. $f = 0$, $g(x) = 0.01x$ if $0 \leqq x \leqq \frac{1}{2}\pi$, $g(x) = 0.01(\pi - x)$ if $\frac{1}{2}\pi < x \leqq \pi$

13. How does doubling the tension affect the pitch of the fundamental tone of a string?

14. How does the frequency of the fundamental mode of the vibrating string depend on the length of the string, the tension, and the mass per unit length?

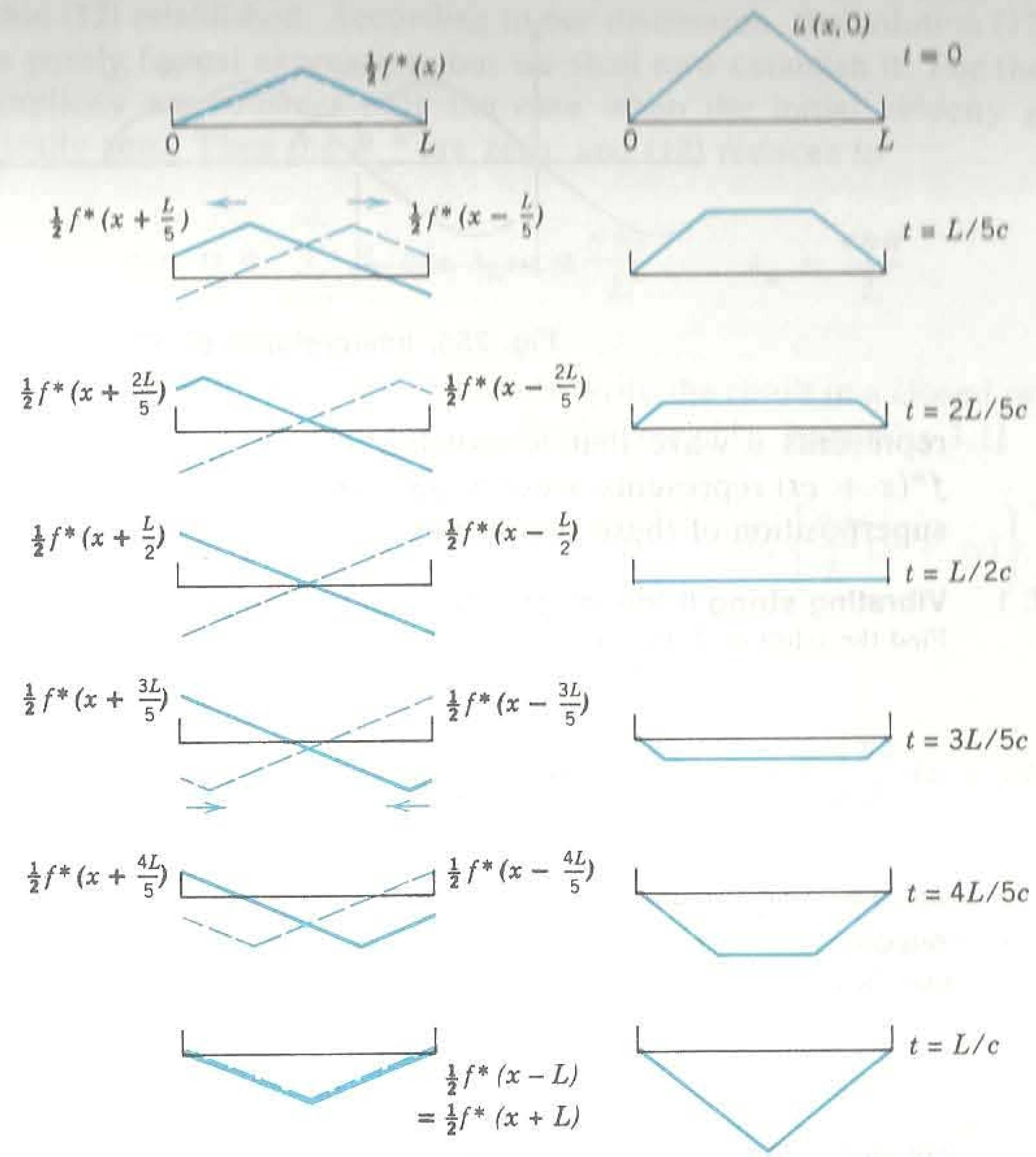


Fig. 256. Solution $u(x, t)$ in Example 1 for various values of t (right part of the figure) obtained as the superposition of a wave traveling to the right (dashed) and a wave traveling to the left (left part of the figure)

15. What is the ratio of the amplitudes of the fundamental mode and the second overtone in Prob. 7? The ratio $a_1^2/(a_1^2 + a_2^2 + \cdots)$? *Hint.* Use Parseval's identity in Sec. 10.8.

Find solutions $u(x, y)$ of the following equations by separating variables.

16. $u_x + u_y = 0$ **17.** $u_x - u_y = 0$ **18.** $xu_x - yu_y = 0$

19. $yu_x - xu_y = 0$ **20.** $u_{xx} + u_{yy} = 0$ **21.** $u_x - yu_y = 0$

22. $u_x + u_y = 2(x + y)u$ **23.** $u_{xy} - u = 0$ **24.** $x^2u_{xy} + 3y^2u = 0$

Forced vibrations of an elastic string

25. Show that forced vibrations of an elastic string are governed by

$$u_{tt} = c^2u_{xx} + \frac{P}{\rho}, \tag{18}$$

where $P(x, t)$ is the external force per unit length acting perpendicular to the string.

26. Assume the external force to be sinusoidal, say, $P = A\rho \sin \omega t$. Show that

$$P/\rho = A \sin \omega t = \sum_{n=1}^{\infty} k_n(t) \sin \frac{n\pi x}{L}$$

where $k_n(t) = (2A/n\pi)(1 - \cos n\pi) \sin \omega t$; consequently $k_n = 0$ (n even), and $k_n = (4A/n\pi) \sin \omega t$ (n odd). Furthermore, show that substitution of

$$u(x, t) = \sum_{n=1}^{\infty} G_n(t) \sin \frac{n\pi x}{L} \quad \text{into (1) gives} \quad \ddot{G}_n + \lambda_n^2 G_n = 0, \quad \lambda_n = \frac{cn\pi}{L}.$$

27. Show that by substituting u and P/ρ from Prob. 26 into (18) we obtain

$$\ddot{G}_n + \lambda_n^2 G_n = \frac{2A}{n\pi}(1 - \cos n\pi) \sin \omega t, \quad \lambda_n = \frac{cn\pi}{L}.$$

Show that if $\lambda_n^2 \neq \omega^2$, the solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t + \frac{2A(1 - \cos n\pi)}{n\pi(\lambda_n^2 - \omega^2)} \sin \omega t.$$

28. Determine B_n and B_n^* in Prob. 27 so that u satisfies the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

29. Show that in the case of resonance ($\lambda_n = \omega$),

$$G_n(t) = B_n \cos \omega t + B_n^* \sin \omega t - \frac{A}{n\pi\omega}(1 - \cos n\pi)t \cos \omega t.$$

30. Show that a problem (1)–(4) with more complicated boundary conditions, say, $u(0, t) = 0$, $u(L, t) = h(t)$, can be reduced to a problem for a new function v satisfying conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f_1(x)$, $v_t(x, 0) = g_1(x)$ but a nonhomogeneous wave equation. *Hint.* Set $u = v + w$ and determine w suitably.

11.4 D'Alembert's Solution of the Wave Equation

It is interesting to note that the solution (17), Sec. 11.3, of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho}, \tag{1}$$

can be immediately obtained by transforming (1) in a suitable way, namely, by introducing the new independent variables[1]

$$v = x + ct, \qquad z = x - ct. \tag{2}$$

[1] We mention that the general theory of partial differential equations provides a systematic way for finding this transformation that will simplify the equation. See Ref. [C9] in Appendix 1.

the boundary conditions (2) in that section the function f must be odd and must have period $2L$.

Our result shows that the two initial conditions and the boundary conditions determine the solution uniquely.

The solution of the wave equation by the Laplace and Fourier transform methods will be shown in Secs. 11.13 and 11.14.

Problem Set 11.4

Using (14), sketch a figure (of the type in Fig. 256, Sec. 11.3) of the deflection $u(x, t)$ of a vibrating string (length $L = 1$, ends fixed, $c = 1$) starting with initial velocity zero and the following initial deflection $f(x)$, where k is small, say, $k = 0.01$.

1. $f(x) = kx(1 - x)$ **2.** $f(x) = k \sin 2\pi x$ **3.** $f(x) = k(x - x^3)$

4. $f(x) = k(x^2 - x^4)$ **5.** $f(x) = k \sin^2 \pi x$ **6.** $f(x) = k(x^3 - x^5)$

7. Show that c is the speed of the two waves given by (4).

8. If a steel wire 2 meters in length weighs 0.8 nt (about 0.18 lb) and is stretched by a tensile force of 200 nt (about 45 lb), what is the corresponding speed c of transverse waves?

9. What are the frequencies of the eigenfunctions in Prob. 8?

10. Solve the equation of a string of length L

$$u_{tt} = c^2 u_{xx} - \gamma^2 u$$

moving in an elastic medium ($\gamma^2 = const$, proportional to the elasticity coefficient of the medium), fixed at the ends and subject to initial displacement $f(x)$ and initial velocity zero.

11. Show that, because of the boundary condition (2) in Sec. 11.3, the function f in (14) of the present section must be odd and of period $2L$.

Using the indicated transformations, solve the following equations.

12. $u_{xy} - u_{yy} = 0 \quad (v = x, z = x + y)$

13. $xu_{xy} = yu_{yy} + u_y \quad (v = x, z = xy)$

14. $u_{xx} - 2u_{xy} + u_{yy} = 0 \quad (v = x, z = x + y)$

15. $u_{xx} + 2u_{xy} + u_{yy} = 0 \quad (v = x, z = x - y)$

16. $u_{xx} + u_{xy} - 2u_{yy} = 0 \quad (v = x + y, z = 2x - y)$

17. $u_{xx} - 4u_{xy} + 3u_{yy} = 0 \quad (v = x + y, z = 3x + y)$

Types and normal forms of linear partial differential equations. An equation of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y) \tag{15}$$

is said to be **elliptic** if $AC - B^2 > 0$, **parabolic** if $AC - B^2 = 0$, and **hyperbolic** if $AC - B^2 < 0$. [Here A, B, C may be functions of x and y, and the type of (15) may be different in different parts of the xy-plane.]

18. Show that

Laplace's equation $u_{xx} + u_{yy} = 0$ **is elliptic**,

the **heat equation** $u_t = c^2 u_{xx}$ is parabolic,

the **wave equation** $u_{tt} = c^2 u_{xx}$ is hyperbolic,

the **Tricomi equation** $yu_{xx} + u_{yy} = 0$ is of mixed type (elliptic in the upper half-plane and hyperbolic in the lower half-plane).

19. If the equation (15) is *hyperbolic*, it can be transformed to the *normal form* $u_{vz} = F^*(v, z, u, u_v, u_z)$ by setting $v = \Phi(x, y)$, $z = \Psi(x, y)$, where $\Phi = const$ and $\Psi = const$ are the solutions $y = y(x)$ of the equation $Ay'^2 - 2By' + C = 0$ (see Ref. [C9]). Show that in the case of the wave equation (1),

$$\Phi = x + ct, \qquad \Psi = x - ct.$$

20. If (15) is *parabolic*, the substitution $v = x$, $z = \Psi(x, y)$, with Ψ defined as in Prob. 19, reduces it to the *normal form* $u_{vv} = F^*(v, z, u, u_v, u_z)$. Verify this result for the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$.

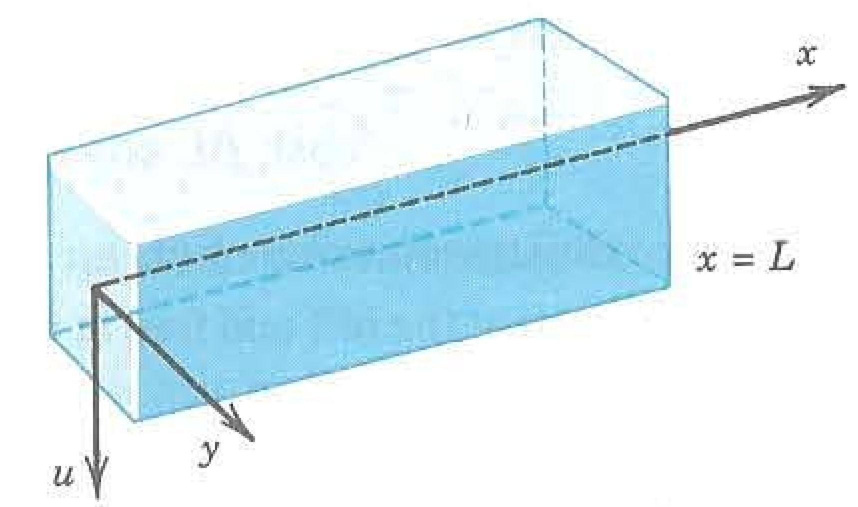


Fig. 257. Undeformed beam in Problem 21

Vibrations of a beam. It can be shown that the small free vertical vibrations of a uniform beam (Fig. 257) are governed by the fourth-order equation

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0 \tag{16}$$ (Ref. [C9].)

where $c^2 = EI/\rho A$ (E = Young's modulus of elasticity, I = moment of inertia of the cross section with respect to the y-axis in the figure, ρ = density, A = cross-sectional area).

21. Substituting $u = F(x)G(y)$ into (16) and separating variables, show that

$$F^{(4)}/F = -\ddot{G}/c^2G = \beta^4 = const,$$

$$F(x) = A \cos \beta x + B \sin \beta x + C \cosh \beta x + D \sinh \beta x,$$

$$G(t) = a \cos c\beta^2 t + b \sin c\beta^2 t.$$

22. Find solutions $u_n = F_n(x)G_n(t)$ of (16) corresponding to zero initial velocity and satisfying the boundary conditions (see Fig. 258)

$u(0, t) = 0$, $u(L, t) = 0$ (ends simply supported for all times t),

$u_{xx}(0, t) = 0$, $u_{xx}(L, t) = 0$ (zero moments, hence zero curvature, at the ends).

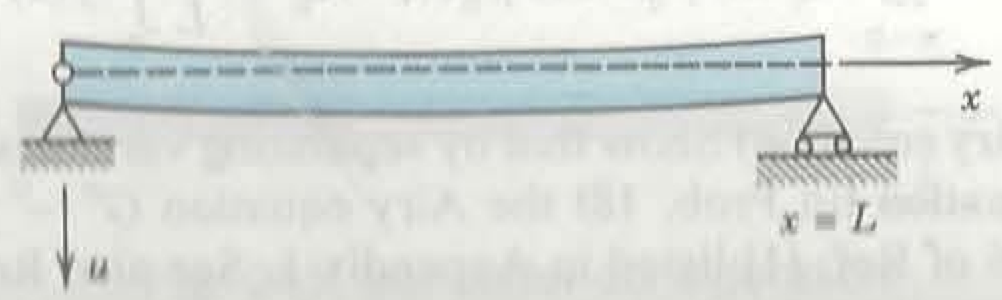


Fig. 258. Beam in Problem 22

Problem Set 11.5

1. Sketch u_1, u_2, u_3 [see (9), with $B_n = 1$, $c = 1$, $L = \pi$] as functions of x for the values $t = 0, 1, 2, 3$. Compare the behavior of these functions.
2. How does the rate of decay of (9) for fixed n depend on the specific heat, the density, and the thermal conductivity of the material?
3. If the first eigenfunction (9) of the bar decreases to half its value within 10 seconds, what is the value of the diffusivity?

Find the temperature $u(x, t)$ in a bar of silver (length 10 cm, constant cross section of area 1 cm^2, density 10.6 gm/cm^3, thermal conductivity 1.04 cal/cm sec °C, specific heat 0.056 cal/gm °C) that is perfectly insulated laterally, whose ends are kept at temperature 0°C and whose initial temperature (in °C) is $f(x)$, where

4. $f(x) = \sin 0.4\pi x$
5. $f(x) = k \sin 0.1\pi x$
6. $f(x) = x$ if $0 < x < 5$ and 0 otherwise
7. $f(x) = 5 - |x - 5|$
8. $f(x) = 0.1x(100 - x^2)$
9. $f(x) = 0.01x(10 - x)$
10. $f(x) = x$ if $0 < x < 2.5$, $f(x) = 2.5$ if $2.5 < x < 7.5$, $f(x) = 10 - x$ if $7.5 < x < 10$

11. Suppose that a bar satisfies the assumptions in the text and that its ends are kept at different constant temperatures $u(0, t) = U_1$ and $u(L, t) = U_2$. Find the temperature $u_I(x)$ in the bar after a long time (theoretically: as $t \to \infty$).
12. In Prob. 11, let the initial temperature be $u(x, 0) = f(x)$. Show that the temperature for any time $t > 0$ is $u(x, t) = u_I(x) + u_{II}(x, t)$ with u_I as before and

$$u_{II} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t},$$

where

$$B_n = \frac{2}{L}\int_0^L [f(x) - u_I(x)] \sin \frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\,dx + \frac{2}{n\pi}[(-1)^n U_2 - U_1].$$

13. **(Insulated ends, adiabatic boundary conditions)** Find the temperature $u(x, t)$ in a bar of length L that is perfectly insulated, also at the ends at $x = 0$ and $x = L$, assuming that $u(x, 0) = f(x)$. *Physical information:* The flux of heat through the faces at the ends is proportional to the values of $\partial u/\partial x$ there. Show that this situation corresponds to the conditions

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0, \qquad u(x, 0) = f(x).$$

Show that the method of separating variables yields the solution

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} e^{-(cn\pi/L)^2 t}$$

where, by (2) in Sec. 10.5,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

14. Find the temperature in the bar in Prob. 13 if the left end is kept at temperature zero, the right end is perfectly insulated, and the initial temperature is $U_0 = const.$

Find the temperature in the bar in Prob. 13 if $L = \pi$, $c = 1$, and

15. $f(x) = 1$
16. $f(x) = x$
17. $f(x) = 0.5 \cos 2x$
18. $f(x) = x^2$
19. $f(x) = x$ if $0 < x < \frac{1}{2}\pi$, $f(x) = \pi - x$ if $\frac{1}{2}\pi < x < \pi$
20. $f(x) = 1$ if $0 < x < \frac{1}{2}\pi$, $f(x) = 0$ if $\frac{1}{2}\pi < x < \pi$
21. $f(x) = x$ if $0 < x < \frac{1}{2}\pi$, $f(x) = 0$ if $\frac{1}{2}\pi < x < \pi$

22. Consider the bar in Probs. 4–10. Assume that the ends are kept at 100°C for a long time. Then at some instant, say, at $t = 0$, the temperature at $x = L$ is suddenly changed to 0°C and kept at this value, while the temperature at $x = 0$ is kept at 100°C. What are the temperatures in the middle of the bar at $t = 1, 2, 3, 10, 50$ sec?
23. **(Radiation at end of bar)** Consider a laterally insulated bar of length π and such that $c = 1$ in (1), whose left end is kept at 0°C and whose right end radiates freely into air of constant temperature 0°C. *Physical information:* The **"radiation boundary condition"** is

$$-u_x(\pi, t) = k[u(\pi, t) - u_0],$$

where $u_0 = 0$ is the temperature of the surrounding air and k is a constant, say, $k = 1$ for simplicity. Show that a solution satisfying these boundary conditions is $u(x, t) = \sin px\, e^{-p^2 t}$, where p is a solution of $\tan p\pi = -p$. Show graphically that this equation has infinitely many positive solutions $p_1, p_2, p_3, \cdots$, where $p_n > n - \frac{1}{2}$ and $\lim_{n\to\infty} (p_n - n + \frac{1}{2}) = 0$.

24. **(Nonhomogeneous heat equation)** Consider the problem consisting of

$$u_t - c^2 u_{xx} = Ne^{-\alpha x}$$

and conditions (2), (3). Here the term on the right may represent loss of heat due to radioactive decay in the bar. Show that this problem may be reduced to a problem for the homogeneous equation by setting $u(x, t) = v(x, t) + w(x)$ and determining $w(x)$ so that v satisfies the homogeneous equation and the conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f(x) - w(x)$.

25. **(Radiation)** If the bar in the text is free to radiate into the surrounding medium kept at temperature zero, the equation becomes

$$v_t = c^2 v_{xx} - \beta v.$$

Show that this equation can be reduced to the form (1) by setting $v(x, t) = u(x, t)w(t)$.

26. Consider $v_t = c^2 v_{xx} - v$ $(0 < x < L,\ t > 0)$, $v(0, t) = 0$, $v(L, t) = 0$, $v(x, 0) = f(x)$, where the term $-v$ corresponds to heat transfer to the surrounding medium kept at temperature zero. Reduce the equation by setting $v(x, t) = u(x, t)w(t)$ with w such that u is given by (10), (11).
27. **(Heat flux)** What is the heat flux $\phi(t) = -Ku_x(0, t)$ across $x = 0$ for the solution (10)? Note that $\phi(t) \to 0$ as $t \to \infty$. Is this physically understandable?
28. Solve (1), (2), (3) with $L = \pi$ and $f(x) = U_0 = const$ if $0 < x < \pi/2$ and

29. If the bar in Prob. 28 consists of two iron parts ($c^2 = 0.16$ Cgs unit) of initial temperatures 20°C and 0°C brought into perfect contact at $t = 0$, what is the approximate temperature at their common face at $t = 10, 20, 30$ seconds? (Use only the first term of the solution.)

30. **(Bar with heat generation)** If heat is generated at a constant rate throughout a bar of length $L = \pi$ with initial temperature $f(x)$ and ends at $x = 0$ and $x = \pi$ kept at temperature zero, the heat equation is $u_t = c^2 u_{xx} + H$ with constant $H > 0$. Solve this problem. *Hint.* Set $u = v - Hx(x - \pi)/2c^2$.

Two-Dimensional Problems

31. **(Laplace's equation)** Find the potential in the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$ if the upper side is kept at the potential $\sin \frac{1}{2}\pi x$ and the other sides are kept at zero.

32. Find the potential in the rectangle $0 \leqq x \leqq 20$, $0 \leqq y \leqq 40$ whose upper side is kept at potential 220 V and whose other sides are grounded.

33. **(Heat flow in a plate)** The faces of a thin square plate (Fig. 264, where $a = 24$) are perfectly insulated. The upper side is kept at temperature 20°C and the other sides are kept at 0°C. Find the steady-state temperature $u(x, y)$ in the plate.

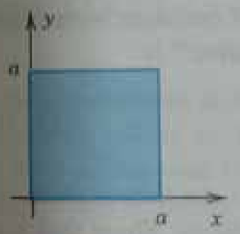


Fig. 264. Square plate

34. Find solutions u of the two-dimensional heat equation

$$u_t = c^2(u_{xx} + u_{yy})$$

in the thin plate in Fig. 264 with $a = \pi$ such that $u = 0$ on the vertical sides, assuming that the faces and the horizontal sides of the plate are perfectly insulated.

35. Find formulas similar to (19), (20), for the temperature distribution in the rectangle R considered in the text when the lower side of R is kept at temperature $f(x)$ and the three other sides are kept at 0.

36. Find the steady-state temperature in the plate in Prob. 33 if the lower side is kept at U_0 °C, the upper at U_1 °C, and the other two sides at 0°C. *Hint.* Split the problem into two problems in which the boundary temperature is zero on three sides for each problem.

37. **(Mixed boundary value problem)** Find the steady-state temperature in the plate in Prob. 33 with ...

SET 11.9

6. Show that $u_n = r^n \cos n\theta$, $u_n = r^n \sin n\theta$, $n = 0, 1, \cdots$, are solutions of $\nabla^2 u = 0$ with $\nabla^2 u$ given by (4).

7. Assuming that termwise differentiation is permissible, show that a solution of the Laplace equation in the disk $R < 1$ satisfying the boundary condition $u(R, \theta) = f(\theta)$ (f given) is

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n \left(\frac{r}{R}\right)^n \cos n\theta + b_n \left(\frac{r}{R}\right)^n \sin n\theta \right]$$

where a_n, b_n are the Fourier coefficients of f (see Sec. 10.3).

Electrostatic potential. Steady-state heat problems. The electrostatic potential u satisfies Laplace's equation $\nabla^2 u = 0$ in any region free of charges. Also, the heat equation $u_t = c^2 \nabla^2 u$ (see Sec. 11.5) reduces to Laplace's equation if the temperature u is independent of time t (**"steady-state case"**). Find the electrostatic potential (equivalently: the steady-state temperature distribution) in the disk $r < 1$ corresponding to the following boundary values.

8. $u(\theta) = 10 \cos^2 \theta$

9. $u(\theta) = 40 \sin^3 \theta$

10. $u(\theta) = \begin{cases} 110 & \text{if } -\pi/2 < \theta < \pi/2 \\ 0 & \text{if } \pi/2 < \theta < 3\pi/2 \end{cases}$

11. $u(\theta) = \begin{cases} -100 & \text{if } -\pi < \theta < 0 \\ 100 & \text{if } 0 < \theta < \pi \end{cases}$

12. $u(\theta) = \begin{cases} \theta & \text{if } -\pi/2 < \theta < \pi/2 \\ \pi - \theta & \text{if } \pi/2 < \theta < 3\pi/2 \end{cases}$

13. $u(\theta) = \begin{cases} \theta & \text{if } -\pi/2 < \theta < \pi/2 \\ 0 & \text{if } \pi/2 < \theta < 3\pi/2 \end{cases}$

14. $u(\theta) = \theta^2 \quad (-\pi < \theta < \pi)$

15. $u(\theta) = |\theta| \quad (-\pi < \theta < \pi)$

16. Find a formula for the potential u on the x-axis in Prob. 15. Use the first four terms of this series for computing u at $x = -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75$ (two decimals).

17. Find a formula for the potential u on the y-axis in Prob. 15.

18. Find the electrostatic potential in the semidisk $r < 1$, $0 < \theta < \pi$, which is equal to $110\theta(\pi - \theta)$ on the semicircle $r = 1$ and 0 on the segment $-1 < x < 1$.

19. Find the steady-state temperature u in a semicircular thin plate $r < a$, $0 < \theta < \pi$, if the semicircle $r = a$ is kept at constant temperature u_0 and the bounding segment $-a < x < a$ is kept at $u = 0$. (Use separation of variables.)

20. **(Laplacian in cylindrical coordinates)** Show that the Laplacian in cylindrical coordinates r, θ, z defined by $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ is

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}.$$

Express $\nabla^2 u = u_{xx} + u_{yy}$ in terms of the coordinates x^*, y^* given by

21. $x^* = ax + b, \quad y^* = cy + d$

22. $x^* = x + y, \quad y^* = x - y$

23. $x^* = x^2, \quad y^* = y^2$

24. $x^* = 1/x, \quad y^* = 1/y$

25. $x^* = x \cos \alpha - y \sin \alpha, \quad y^* = x \sin \alpha + y \cos \alpha$ (*"rotation* through α")

26. **(Neumann problem)** Show that the solution of the Neumann problem $\nabla^2 u = 0$ if $r < R$, $u_n(R, \theta) = f(\theta)$ (n the outer normal) is

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with arbitrary A_0 and

$$A_n = \frac{2}{\pi n R^{n-1}} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \qquad B_n = \frac{2}{\pi n R^{n-1}} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta.$$

27. Show that (9), Sec. 9.4, imposes on $f(\theta)$ in Prob. 26 the condition

$$\int_{-\pi}^{\pi} f(\theta) \, d\theta = 0,$$

usually called a *"compatibility condition."*

28. **(Neumann problem)** Solve $\nabla^2 u = 0$ in the annulus $1 < r < 3$ if $u_r(1, \theta) = \sin \theta$, $u_r(3, \theta) = 0$.

11.10 Circular Membrane. Use of Fourier–Bessel Series

Circular membranes occur in drums, pumps, microphones, telephones, and so on, and this accounts for their great importance in engineering. Whenever a circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes!), its vibrations are governed by the two-dimensional wave equation (3′), Sec. 11.7, which we now write in polar coordinates defined by $x = r \cos \theta$, $y = r \sin \theta$ in the form [see (4) in the last section]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$

Figure 272 shows our membrane of radius R, for which we shall determine solutions $u(r, t)$ that are radially symmetric,[9] that is, do not depend on θ. Then the wave equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right). \tag{1}$$

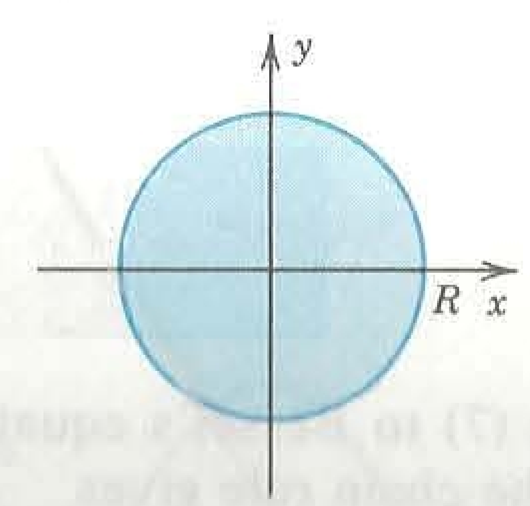


Fig. 272. Circular membrane

$u(r, \phi) = \frac{\ }{r} + \frac{\ }{2r^2} P_1(\cos \phi) - \frac{\ }{8r^4} P_3(\cos \phi) + \cdots .$

Partial sums of these series can now be used for computing approximate values of the potential. Also, it is interesting to see that far away from the sphere the potential is approximately that of a point charge, namely, $55/r$. (Compare with Example 3 in Sec. 8.9.) ■

Problem Set 11.12

1. Verify by substitution that $u_n(r, \phi)$ and $u_n^*(r, \phi)$, $n = 0, 1, 2$, in (8*) are solutions of (2).
2. Find the surfaces on which the functions u_1, u_2, u_3 are zero.
3. Sketch the functions $P_n(\cos \phi)$ for $n = 0, 1, 2$, [see (11′), Sec. 5.3].
4. Sketch the functions $P_3(\cos \phi)$ and $P_4(\cos \phi)$.

Let r, θ, ϕ be the spherical coordinates used in the text. Find the potential in the interior of the sphere $R = 1$, assuming that there are no charges in the interior and the potential on the surface is $f(\phi)$, where

5. $f(\phi) = 1$
6. $f(\phi) = \cos \phi$
7. $f(\phi) = \cos 2\phi$
8. $f(\phi) = 1 - \cos^2 \phi$
9. $f(\phi) = \cos^3 \phi$
10. $f(\phi) = \cos 3\phi + 3 \cos \phi$
11. $f(\phi) = 10 \cos^3 \phi - 3 \cos^2 \phi - 5 \cos \phi - 1$

12. Show that in Prob. 5, the potential exterior to the sphere is the same as that of a point charge at the origin.
13. Sketch the intersections of the equipotential surfaces in Prob. 6 with the xz-plane.
14. Find the potential exterior to the sphere in Probs. 5–11.
15. Derive the values of A_0, A_1, A_2, A_3 in Example 1 from (13).
16. In Example 1, sketch the sum of the three explicitly given terms for $r = 1$ and see how well this sum approximates the given boundary function.
17. Find the temperature in a homogeneous ball of radius 1 if its lower boundary hemisphere is kept at 0°C and its upper at 20°C.
18. Show that $P'_{n+1}(x) - P'_{n-1}(x) = (2n + 1)P_n(x)$. (Use Prob. 8, Sec. 5.3.)
19. Show that $\int_0^1 P_n(x)\,dx = [P_{n-1}(0) - P_{n+1}(0)]/(2n + 1)$. (Use Prob. 18 and Prob. 12, Sec. 5.3.) Using this, verify A_1, A_2, A_3 in Example 1 and compute A_5.

20. **(Transmission line equations)** Consider a long cable or telephone wire (Fig. 279) that is imperfectly insulated so that leaks occur along the entire length of the cable. The source S of the current $i(x, t)$ in the cable is at $x = 0$, the receiving end T at $x = l$. The current flows from S to T, through the load, and returns to the ground. Let the constants R, L, C, and G denote the resistance, inductance, capacitance to ground, and conductance to ground, respectively, of the cable per unit length. Show that

$$-\frac{\partial u}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \textbf{(First transmission line equation)}$$

where $u(x, t)$ is the potential in the cable. *Hint.* Apply Kirchhoff's voltage law to a small portion of the cable between x and $x + \Delta x$ (difference of the potentials at x and $x + \Delta x$ = resistive drop + inductive drop).

21. Show that for the cable in Prob. 20,

$$-\frac{\partial i}{\partial x} = Gu + C\frac{\partial u}{\partial t} \qquad \textbf{(Second transmission line equation).}$$

Hint. Use Kirchhoff's current law (difference of the currents at x and $x + \Delta x$ = loss due to leakage to ground + capacitive loss).

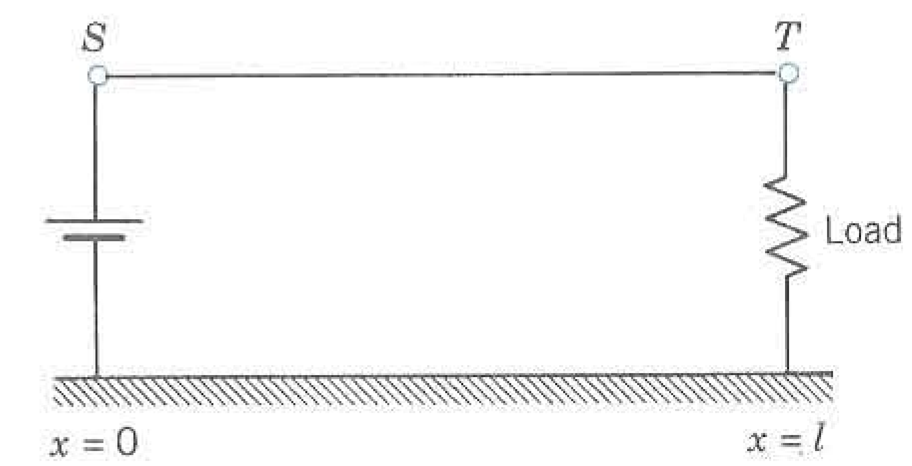


Fig. 279. Transmission line

22. Show that elimination of i or u from the transmission line equations leads to

$$u_{xx} = LCu_{tt} + (RC + GL)u_t + RGu,$$

$$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi.$$

23. **(Telegraph equations)** For a submarine cable, G is negligible and the frequencies are low. Show that this leads to the so-called *submarine cable equations* or **telegraph equations**

$$u_{xx} = RCu_t, \qquad i_{xx} = RCi_t.$$

24. Find the potential in a submarine cable with ends ($x = 0$, $x = l$) grounded and initial voltage distribution $U_0 = const.$

25. **(High-frequency line equations)** Show that in the case of alternating currents of high frequencies the equations in Prob. 22 can be approximated by the so-called **high-frequency line equations**

$$u_{xx} = LCu_{tt}, \qquad i_{xx} = LCi_{tt}.$$

Solve the first of them, assuming that the initial potential is $U_0 \sin(\pi x/l)$, $u_t(x, 0) = 0$ and $u = 0$ at the ends $x = 0$ and $x = l$ for all t.

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2}, \qquad \text{thus} \qquad \frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2} W = 0.$$

Since this equation contains only a derivative with respect to x, it may be regarded as an ordinary differential equation for $W(x, s)$ considered as a function of x. A general solution is

$$W(x, s) = A(s)e^{sx/c} + B(s)e^{-sx/c}. \tag{9}$$

From (6) we obtain, writing $F(s) = \mathcal{L}\{f(t)\}$,

$$W(0, s) = \mathcal{L}\{w(0, t)\} = \mathcal{L}\{f(t)\} = F(s)$$

and, assuming that the order of integrating with respect to t and taking the limit as $x \to \infty$ can be interchanged,

$$\lim_{x\to\infty} W(x, s) = \lim_{x\to\infty} \int_0^\infty e^{-st} w(x, t)\, dt = \int_0^\infty e^{-st} \lim_{x\to\infty} w(x, t)\, dt = 0.$$

This implies $A(s) = 0$ in (9) because $c > 0$, so that for every fixed positive s the function $e^{sx/c}$ increases as x increases. Note that we may assume $s > 0$ since a Laplace transform generally exists for *all* s greater than some fixed γ (Sec. 6.2). Hence we have

$$W(0, s) = B(s) = F(s),$$

so that (9) becomes

$$W(x, s) = F(s)e^{-sx/c}.$$

From the second shifting theorem (Sec. 6.3) with $a = x/c$ we obtain the inverse transform

$$w(x, t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right) \qquad \text{(Fig. 281)}, \tag{10}$$

that is,

$$w(x, t) = \sin\left(t - \frac{x}{c}\right) \quad \text{if} \quad \frac{x}{c} < t < \frac{x}{c} + 2\pi \quad \text{or} \quad ct > x > (t - 2\pi)c$$

and zero otherwise. This is a single sine wave traveling to the right with speed c. Note that a point x remains at rest until $t = x/c$, the time needed to reach that x if one starts at $t = 0$ (start of the motion of the left end) and travels with speed c. The result agrees with our physical intuition. Since we proceeded formally, we must verify that (10) satisfies the given conditions. We leave this to the student. ■

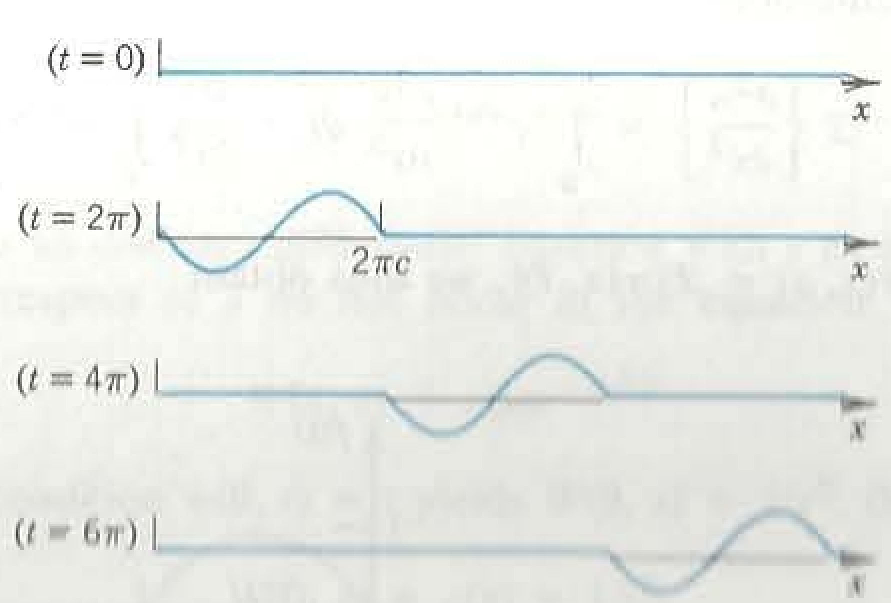


Fig. 281. Traveling wave in Example 2

Problem Set 11.13

1. Sketch a figure similar to Fig. 281 if $c = 1$ and f is "triangular" as in Example 1, Sec. 11.3, with $k = L/2 = 1$.
2. How does the speed of the wave in Example 2 depend on the tension and the mass of the string?
3. Verify the solution in Example 2. What traveling wave do we obtain in Example 2 if we impose a (nonterminating) sinusoidal motion of the left end starting at $t = 0$?

Solve by Laplace transforms:

4. $\dfrac{\partial u}{\partial x} + 2x \dfrac{\partial u}{\partial t} = 2x,\ u(x, 0) = 1,\ u(0, t) = 1$
5. $x \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = xt,\ u(x, 0) = 0 \text{ if } x \geqq 0,\ u(0, t) = 0 \text{ if } t \geqq 0.$
6. Solve Prob. 5 by another method.

Find the temperature $w(x, t)$ in a semi-infinite laterally insulated bar extending from $x = 0$ along the x-axis to ∞, assuming that the initial temperature is 0, $w(x, t) \to 0$ as $x \to \infty$ for every fixed $t \geqq 0$, and $w(0, t) = f(t)$. Proceed as follows.

7. Set up the model and show that the Laplace transform leads to

$$sW(x, s) = c^2 \frac{\partial^2 W}{\partial x^2}, \qquad W = \mathcal{L}\{w\},$$

and

$$W(x, s) = F(s)e^{-\sqrt{s}x/c}, \qquad F = \mathcal{L}\{f\}.$$

8. Applying the convolution theorem in Prob. 7, show that

$$w(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t f(t - \tau)\tau^{-3/2} e^{-x^2/4c^2\tau}\, d\tau.$$

9. Let $w(0, t) = f(t) = u(t)$ (Sec. 6.3). Denote the corresponding w, W, and F by w_0, W_0, and F_0. Show that then in Prob. 8,

$$w_0(x, t) = \frac{x}{2c\sqrt{\pi}} \int_0^t \tau^{-3/2} e^{-x^2/4c^2\tau}\, d\tau = 1 - \operatorname{erf}\left(\frac{x}{2c\sqrt{t}}\right)$$

with the error function erf as defined in Problem Set 11.6.

10. **(Duhamel's formula**[14]**)** Show that in Prob. 9,

$$W_0(x, s) = \frac{1}{s} e^{-\sqrt{s}x/c}$$

and the convolution theorem gives *Duhamel's formula*

$$w(x, t) = \int_0^t f(t - \tau) \frac{\partial w_0}{\partial \tau}\, d\tau.$$

[14] JEAN MARIE CONSTANT DUHAMEL (1797—1872), French mathematician.

1. $F = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{N}\sin Nx\right]$ (N odd)

5. $F = 2\left(\sin x - \frac{1}{2}\sin 2x + \cdots + \frac{(-1)^{N+1}}{N}\sin Nx\right)$,
$E^* = 8, 5, 3.6, 2.8, 2.3$

7. $F = \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \cdots + \frac{(-1)^{N+1}}{N^2}\cos Nx\right)$,
$E^* \approx 4.14, 1.00, 0.38, 0.18, 0.10$

9. $F = \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{9\pi}\sin 3x - \frac{1}{4}\sin 4x + \frac{2}{25\pi}\sin 5x + \cdots$,
$E^* = \frac{\pi^3}{12} - \pi\left[\frac{4}{\pi^2} + \frac{1}{4} + \frac{4}{81\pi^2} - \frac{1}{16} + \frac{4}{625\pi^2} + \cdots\right]$; 1.311, 0.525, 0.509, 0.313, 0.311

15. Use the Fourier series $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$.

PROBLEM SET 10.9, page 605

9. $\frac{2}{\pi}\int_0^\infty \left[\left(1 - \frac{2}{w^2}\right)\sin w + \frac{2}{w}\cos w\right]\frac{\cos wx}{w}\,dw$

11. $\frac{2}{\pi}\int_0^\infty \left[\frac{a \sin aw}{w} + \frac{\cos aw - 1}{w^2}\right]\cos xw\,dw$

13. $A = \frac{2}{\pi}\int_0^\infty \frac{\cos wv}{1 + v^2}\,dv = e^{-w}\ (w > 0)$, $f(x) = \int_0^\infty e^{-w}\cos wx\,dw$

15. $f(ax) = \int_0^\infty A(w)\cos axw\,dw = \int_0^\infty A\left(\frac{p}{a}\right)\cos xp\,\frac{dp}{a}$, where $wa = p$.
If we write again w instead of p, the result follows.

17. Differentiating (10) we have $\frac{d^2A}{dw^2} = -\frac{2}{\pi}\int_0^\infty f^*(v)\cos wv\,dv$, $f^*(v) = v^2 f(v)$, and the result follows.

PROBLEM SET 10.10, page 610

1. $\sqrt{2/\pi}\,(\sin 2w - 2\sin w)/w$
3. $\sqrt{2/\pi}\,(aw \sin aw + \cos aw - 1)/w^2$
7. $e^{-w}\sqrt{\pi/2}$
9. $\sqrt{2/\pi}\,[(2 - w^2)\cos w + 2w \sin w - 2]/w^3$
11. $\sqrt{\pi/2}$ if $0 < w < \pi$, 0 if $w > \pi$
13. $\sqrt{\pi/2}\cos w$ if $|w| < \pi/2$, 0 if $|w| > \pi/2$
17. $\sqrt{\pi/2}\,e^{-w}\cos w$
19. No

PROBLEM SET 10.11, page 618

1. $1/(1 + iw)\sqrt{2\pi}$
3. $\sqrt{2/\pi}\,(2 - w)^{-1}\sin(2 - w)$
5. $[-1 + (1 + iaw)e^{-iaw}]/w^2\sqrt{2\pi}$
7. $i\sqrt{2/\pi}\,(\cos w - 1)/w$

17. $\frac{1}{2} - \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

19. $\frac{2}{\pi}\left(\sin x - \frac{2}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x - \frac{2}{6}\sin 6x + \cdots\right)$

21. $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

23. $-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x - + \cdots$

25. $\frac{8}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$

27. $\frac{\pi^2}{12} - \frac{2}{\pi}\cos x - \frac{1}{2^2}\cos 2x + \frac{2}{3^3\pi}\cos 3x + \frac{1}{4^2}\cos 4x - \cdots$

29. $\frac{8}{\pi}\left(\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots\right)$

31. $-\frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}5\pi x + \cdots\right)$

33. $\frac{4}{\pi}\left(\sin\frac{\pi}{2}x - \frac{1}{2}\sin \pi x + \frac{1}{3}\sin\frac{3\pi}{2}x - + \cdots\right)$

35. $\frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) - \frac{1}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

37. $-\frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9}\cos 3\pi x + \cdots\right) + \frac{2}{\pi}\left(2\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

39. $-\frac{1}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$
$+ \frac{2}{\pi}\left(\sin \pi x - \frac{1}{2}\sin 2\pi x + - \cdots\right)$

41. $\pi/4$
43. $\pi^3/32$
47. 5.168, 0.075, 0.075, 0.012, 0.012, 0.004

49. $y = C_1\cos \omega t + C_2\sin \omega t + \frac{\pi^2}{12\omega^2} - \frac{1}{\omega^2 - 1}\cos t + \frac{1}{4(\omega^2 - 4)}\cos 2t - + \cdots$

PROBLEM SET 11.1, page 628

25. $u = f(x)$
27. $u_x = f(y)$, $u = xf(y) + g(y)$
29. $u = c(y)e^{x^2y}$
31. $u = v(x) + w(y)$
33. $u = c = const$
35. $u = cx + g(y)$

PROBLEM SET 11.3, page 637

1. $u = 0.02\cos t \sin x$
3. $u = k(\cos t \sin x - \cos 2t \sin 2x)$

5. $u = \frac{4}{5\pi}\left(\frac{1}{4}\cos 2t \sin 2x - \frac{1}{36}\cos 6t \sin 6x + \frac{1}{100}\cos 10t \sin 10x - + \cdots\right)$

7. $u = \frac{8k}{\pi}\left(\cos t \sin x + \frac{1}{3^3}\cos 3t \sin 3x + \frac{1}{5^3}\cos 5t \sin 5x + \cdots\right)$

9. $u = 12k\left[\left(\frac{\pi}{1^3} - \frac{8}{1^5\pi}\right)\cos t \sin x + \left(\frac{\pi}{3^3} - \frac{8}{3^5\pi}\right)\cos 3t \sin 3x + \cdots\right]$

11. $u = 0.1 \sin x\,(\cos t - 2 \sin t)$

15. $27{,}960/\pi^6 \approx 0.9986$

17. $u = ke^{c(x+y)}$

19. $u = k \exp[c(x^2 + y^2)]$

21. $u = ky^c e^{cx}$

23. $u = k \exp(cx + y/c)$

PROBLEM SET 11.4, page 642

9. $17.5n$ cycles/sec

13. $u = f_1(x) + f_2(xy)$

15. $u = xf_1(x - y) + f_2(x - y)$

17. $u = f_1(x + y) + f_2(3x + y)$

23. $u = \frac{8L^2}{\pi^3}\left(\cos c\left(\frac{\pi}{L}\right)^2 t \sin\frac{\pi x}{L} + \frac{1}{3^3}\cos c\left(\frac{3\pi}{L}\right)^2 t \sin\frac{3\pi x}{L} + \cdots\right)$

25. $u(0, t) = 0,\ u(L, t) = 0,\ u_x(0, t) = 0,\ u_x(L, t) = 0$

27. $\beta L \approx \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \cdots$ (more exactly 4.730, 7.853, 10.996, $\cdots$)

29. $\beta L \approx \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \cdots$ (more exactly 1.875, 4.694, 7.855, $\cdots$)

PROBLEM SET 11.5, page 654

3. $\lambda_1^2 = (\ln 2)/10,\ c^2 = 0.00702L^2$

5. $u = \sin 0.1\pi x\, e^{-1.752\pi^2 t/100}$

7. $u = \frac{40}{\pi^2}\left(\sin 0.1\pi x\, e^{-0.01752\pi^2 t} - \frac{1}{9}\sin 0.3\pi x\, e^{-0.01752(3\pi)^2 t} + - \cdots\right)$

9. $u = \frac{8}{\pi^3}\left(\sin 0.1\pi x\, e^{-0.01752\pi^2 t} + \frac{1}{3^3}\sin 0.3\pi x\, e^{-0.01752(3\pi)^2 t} + \cdots\right)$

11. Since the temperatures at the ends are kept constant, the temperature will approach a steady-state (time-independent) distribution $u_I(x)$ as $t \to \infty$, and $u_I = U_1 + (U_2 - U_1)x/L$, the solution of (1) with $\partial u/\partial t = 0$ satisfying the boundary conditions.

15. $u = 1$

17. $u = 0.5 \cos 2x\, e^{-4t}$

19. $u = \frac{\pi}{4} - \frac{8}{\pi}\left(\frac{1}{4}\cos 2x\, e^{-4t} + \frac{1}{36}\cos 6x\, e^{-36t} + \cdots\right)$

21. $u = \frac{\pi}{8} + \left(1 - \frac{2}{\pi}\right)\cos x\, e^{-t} - \frac{1}{\pi}\cos 2x\, e^{-4t} - \left(\frac{1}{3} + \frac{2}{9\pi}\right)\cos 3x\, e^{-9t} + \cdots$

25. $w = e^{-\beta t}$

27. $-\frac{K\pi}{L}\sum_{n=1}^{\infty} nB_n e^{-\lambda_n^2 t}$

29. 2.57, 0.52, 0.10°C

31. $u = (\sin\frac{1}{2}\pi x \sinh\frac{1}{2}\pi y)/\sinh\pi$

33. $u = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{24}\,\frac{\sinh[(2n-1)\pi y/24]}{\sinh(2n-1)\pi}$

35. $u(x, y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b - y)}{a}$,

$$A_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

37. $u(x, y) = \frac{A_0}{24}x + \sum_{n=1}^{\infty} A_n \frac{\sinh(n\pi x/24)}{\sinh n\pi}\cos\frac{n\pi y}{24}$,

$$A_0 = \frac{1}{24}\int_0^{24} f(y)\,dy,\quad A_n = \frac{1}{12}\int_0^{24} f(y)\cos\frac{n\pi y}{24}\,dy$$

39. $u = \sum_{n=0}^{\infty} A_n \cos nx\, e^{-ny},\quad A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$,

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,\quad n = 1, 2, \cdots$$

PROBLEM SET 11.6, page 660

7. $\frac{1}{\sqrt{\pi}}\int_{(a-x)/\tau}^{(b-x)/\tau} e^{-w^2}\,dw - \frac{1}{\sqrt{\pi}}\int_{(a+x)/\tau}^{(b+x)/\tau} e^{-w^2}\,dw$

11. $A(p) = \frac{2}{\pi(1 + p^2)},\ B(p) = 0,\ u = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1 + p^2}\cos px\, e^{-c^2p^2t}\,dp$

13. $A(p) = \frac{2\sin p}{\pi p},\ B(p) = 0,\ u = \frac{2}{\pi}\int_0^{\infty}\frac{\sin p}{p}\cos px\, e^{-c^2p^2t}\,dp$

PROBLEM SET 11.8, page 671

1. c increases and so does the frequency.

5. $c\pi\sqrt{260}$ (corresponding eigenfunctions $F_{4,16}$, $F_{16,14}$), etc.

7. $A = ab,\ b = A/a,\ (ma^{-2} + na^2A^{-2})' = 0$ gives $a^2/b^2 = m/n$.

9. $f_1(x) = 2(4x - x^2),\ f_2(y) = 2y - y^2$

11. $B_{mn} = (-1)^{m+1}8/mn\pi$ (n odd), 0 (n even)

13. $4(\cos m\pi/2 - (-1)^m)(\cos n\pi/2 - (-1)^n)/mn\pi^2$

15. $B_{mn} = (-1)^{m+n}ab/mn\pi^2$

17. $B_{mn} = 4[1 - (-1)^n(b + 1)][1 - (-1)^m(a + 1)]/mn\pi^2$

19. $B_{mn} = (-1)^{m+n}\frac{144a^3b^3}{m^3n^3\pi^6}$

21. $u = k\cos\pi\sqrt{5}t \sin\pi x \sin 2\pi y$

23. $u = k\cos 5\pi t \sin 3\pi x \sin 4\pi y$

PROBLEM SET 11.9, page 673

9. $u = 30r\sin\theta - 10r^3\sin 3\theta$

11. $u = \frac{400}{\pi}\left(r\sin\theta + \frac{1}{3}r^3\sin 3\theta + \frac{1}{5}r^5\sin 5\theta + \cdots\right)$

13. $u = \frac{2}{\pi}r\sin\theta + \frac{1}{2}r^2\sin 2\theta - \frac{2}{9\pi}r^3\sin 3\theta - \frac{1}{4}r^4\sin 4\theta + \cdots$

15. $u = \frac{\pi}{2} - \frac{4}{\pi}\left(r\cos\theta + \frac{1}{9}r^3\cos 3\theta + \frac{1}{25}r^5\cos 5\theta + \cdots\right)$

17. $u = \pi/2$

19. $u = \dfrac{4u_0}{\pi}\left(\dfrac{r}{a}\sin\theta + \dfrac{1}{3a^3}r^3\sin 3\theta + \dfrac{1}{5a^5}r^5\sin 5\theta + \cdots\right)$

21. $a^2u_{x^*x^*} + c^2u_{y^*y^*}$

23. $4x^*u_{x^*x^*} + 4y^*u_{y^*y^*} + 2u_{x^*} + 2u_{y^*}$

25. $u_{x^*x^*} + u_{y^*y^*}$

27. Use $\nabla^2 u = 0$ and $u_n = u_r$.

PROBLEM SET 11.10, page 680

5. $T = 6.828\rho R^2 f_1^{\,2}$, f_1 the fundamental frequency

9. No

15. $u = 4k \sum_{m=1}^{\infty} \dfrac{J_2(\alpha_m)}{\alpha_m^{\,2} J_1^{\,2}(\alpha_m)} \cos \alpha_m t \, J_0(\alpha_m r)$

23. $\alpha_{11}/2\pi \approx 0.6099$ (see Table A1 in Appendix 5)

PROBLEM SET 11.11, page 684

3. $u = 160/r + 30$

5. $u = -40 \ln r/(\ln 2) + 150$

17. $u = (u_1 - u_0)(\ln r)/\ln (r_1/r_0) + (u_0 \ln r_1 - u_1 \ln r_0) / \ln (r_1/r_0)$

PROBLEM SET 11.12, page 690

5. $u = 1$

7. $\cos 2\phi = 2\cos^2\phi - 1$, $2x^2 - 1 = \frac{4}{3}P_2(x) - \frac{1}{3}$, $u = \frac{4}{3}r^2P_2(\cos\phi) - \frac{1}{3}$

9. $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$, $u = \frac{2}{5}r^3P_3(\cos\phi) + \frac{3}{5}rP_1(\cos\phi)$

11. $u = 4r^3P_3(\cos\phi) - 2r^2P_2(\cos\phi) + rP_1(\cos\phi) - 2$

17. This is the analog of Example 1 with 55 replaced by 10.

19. $55(3/8 + 5/16) \approx 37.8$

25. $u = U_0 \cos(\pi t/l\sqrt{LC}) \sin(\pi x/l)$

PROBLEM SET 11.13, page 695

5. $U(x, s) = \dfrac{c(s)}{x^s} + \dfrac{x}{s^2(s+1)}$, $\quad U(0, s) = 0, \quad c(s) = 0,$
$u(x, t) = x(t - 1 + e^{-t})$

9. Set $x^2/4c^2\tau = z^2$. Use z as a new variable of integration. Use $\operatorname{erf}(\infty) = 1$.

CHAPTER 11 (REVIEW QUESTIONS AND PROBLEMS), page 700

21. $u = A(x)\cos 4y + B(x)\sin 4y$

23. $u = A(y)e^{-2x} + B(y)e^{x} - 5$

25. $u = g(x)(1 - e^{-y}) + f(x)$

27. $u = f_1(y) + f_2(x + y)$

29. $u = yf_1(x + y) + f_2(x + y)$

31. $u = f_1(x + y) + f_2(2y - x)$

47. $u = \dfrac{200}{\pi^2}\left(\sin\dfrac{\pi x}{50}e^{-0.004572t} - \dfrac{1}{9}\sin\dfrac{3\pi x}{50}e^{-0.04115t} + \cdots\right)$

49. $u = 95\cos 2x\, e^{-4t}$

59. $u = 275/r - 27.5$

PROBLEM SET 12.1, page 711

3. $32 - 24i$

5. $-\frac{7}{41} + \frac{22}{41}i$

7. $-47.2 - 23i$

9. $-10 - 24i$

11. $31/50$

13. $2xy/(x^2 + y^2)$

15. $x^2 - y^2$, x^2

17. 16

PROBLEM SET 12.2, page 717

3. 2.5

5. 1

7. 1

9. 8/17

11. $\sqrt{2}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

13. $10(\cos 0.927 + i\sin 0.927)$

15. $\frac{1}{4}(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

17. $0.563(\cos 0.308 + i\sin 0.308)$

19. -3.042

21. $\pi/4$

23. $-2 + 2i$

25. $-0.227 - 0.974i$

27. $\pm(2 - 2i)$

29. $\pm 1, \pm i, \pm(1 \pm i)/\sqrt{2}$

31. $\pm(1 \pm i)/\sqrt{2}$

33. $\sqrt[6]{2}\left(\cos\dfrac{k\pi}{12} + i\sin\dfrac{k\pi}{12}\right)$, $k = 1, 9, 17$

35. $3 + 2i$, $2 - i$

37. $|z| = \sqrt{x^2 + y^2} \geqq |x|$, etc.

39. Equation (5) holds when $z_1 + z_2 = 0$. Let $z_1 + z_2 \neq 0$ and $c = a + ib = z_1/(z_1 + z_2)$. By (19) in Prob. 37, $|a| \leqq |c|$, $|a - 1| \leqq |c - 1|$. Thus $|a| + |a - 1| \leqq |c| + |c - 1|$. Clearly $|a| + |a - 1| \geqq 1$. Together we have the inequality below; multiply by $|z_1 + z_2|$ to get (5).

$$1 \leqq |c| + |c - 1| = \left|\frac{z_1}{z_1 + z_2}\right| + \left|\frac{z_2}{z_1 + z_2}\right|$$

PROBLEM SET 12.3, page 720

1. Circle, radius 4, center $4i$

3. Annulus with center a

5. Vertical infinite strip

7. Right half-plane

9. Region between the two branches of the hyperbola $xy = 1$

11. Circle $(x - 17/15)^2 + y^2 = (8/15)^2$

PROBLEM SET 12.4, page 725

1. $14 + 8i$, $-1 - 2i$, $4 - 12i$

3. $(9 - 13i)/500$, $-i$, $(-2 - 11i)/1000$

5. $2(x^3 - 3xy^2) - 3x$, $2(3x^2y - y^3) - 3y$

7. $|w| > 9$

9. $|\arg w| \leqq 3\pi/4$

11. $\operatorname{Re}(z^2)/|z^2| = (x^2 - y^2)/(x^2 + y^2) = 1$ if $y = 0$ and -1 if $x = 0$. *Ans.* No.

13. $6z(z^2 + i)^2$

15. $2i/(1 - z)^3$

17. 0

19. $i/2$

21. $-1/27$

23. $-\frac{1}{8}(1 + i)$