

Inverse Laplace transforms via residue theory

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The Laplace transform:

For a function (signal) $f(t)$ which is zero for $t < 0$, the Laplace transform is

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Here we use $s = \sigma + j\omega$ in place of $z = x + jy$, so we have

$$F(s) = \int_0^{\infty} (f(t)e^{-\sigma t})e^{-j\omega t} dt$$

Problem: Given $F(s)$ how do we obtain $f(t)$?

The Fourier transform:

The Fourier transform of $f(t)$ is

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

and the inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{j\omega t} dt$$

In the Laplace transform let,

$$\begin{aligned} \phi(t) &= f(t)e^{-\sigma t}, \quad t \geq 0 \\ &= 0, \quad t < 0 \end{aligned}$$

where σ is a constant. Taking the Fourier transform of $\phi(t)$:

$$\hat{\phi}(\omega) = \int_0^{\infty} \phi(t)e^{-j\omega t} dt = \int_0^{\infty} f(t)e^{-\sigma t}e^{-j\omega t} dt = F(\sigma + j\omega)$$

So, taking the inverse Fourier transform:

$$\phi(t) = f(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t} d\omega$$

Hence,

$$f(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}e^{\sigma t} d\omega$$

Let $s = \sigma + j\omega$, $ds = jd\omega$, then

$$\begin{aligned} f(t) &= \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds, \quad t \geq 0 \\ &= 0, \quad t < 0 \end{aligned}$$

The inverse Laplace transform

The formula for the inverse Laplace transform was obtained in the previous section as:

$$f(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

The relevant questions here are:

1. How do we choose the real parameter σ ?
2. How do we evaluate the integral?

We already know that $f(t) = 0, t < 0$, and we shall see that this gives us an answer to (1).

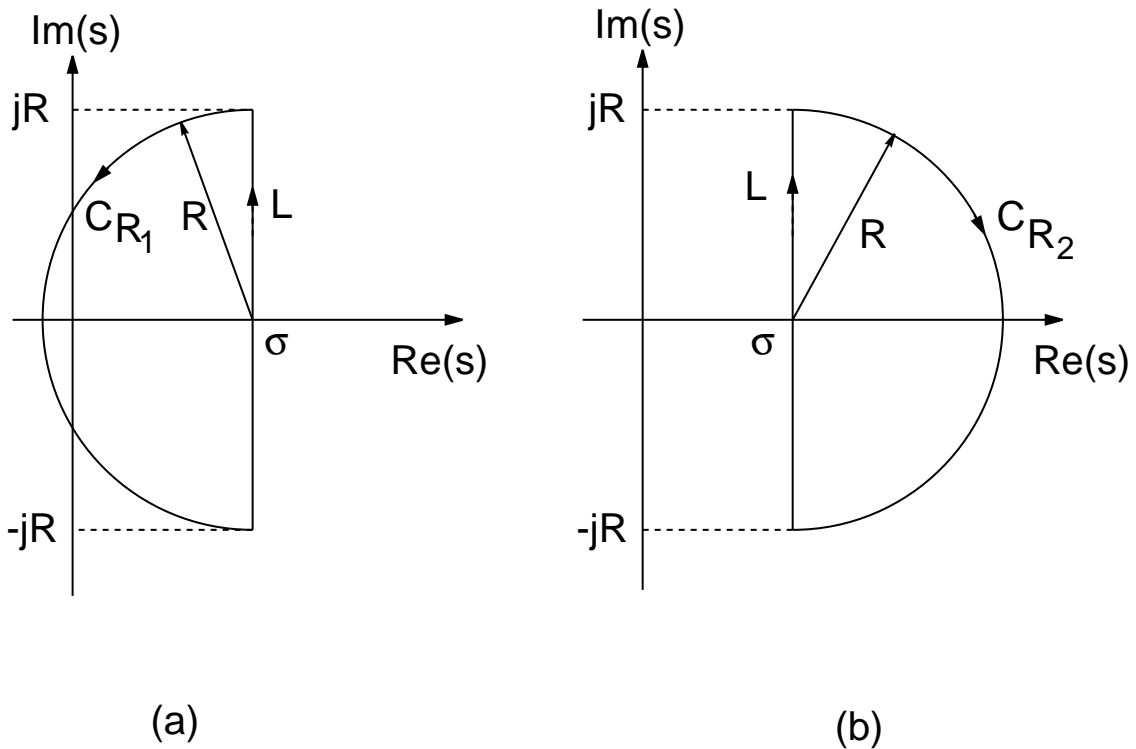


Figure 1: Integration contours

First, consider the closed contour $C_1 = C_{R_1} + L$ shown in Figure 1a. Then,

$$\begin{aligned} \oint_{C_1} F(s)e^{st} ds &= \int_L F(s)e^{st} ds + \int_{C_{R_1}} F(s)e^{st} ds \\ &= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st} ds + \int_{C_{R_1}} F(s)e^{st} ds \\ &= 2\pi j \sum_{\text{poles in } C_1} \text{Res}[F(s)e^{st}] \end{aligned}$$

Note that the residues are at the poles inside C_1 . Now, as $R \rightarrow \infty$, $\int_{\sigma-jR}^{\sigma+jR} F(s)e^{st} ds$ becomes the integral we require, and we can show (Jordan's lemma, see appendix) that for $t > 0$, $\int_{C_{R_1}} F(s)e^{st} ds \rightarrow 0$

as $R \rightarrow \infty$. So, for $t > 0$,

$$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = \sum_{\text{poles left of } \sigma} \text{Res}[F(s)e^{st}]$$

Next consider the contour $C_2 = C_{R_2} + L$ shown in Figure 1b. As before,

$$\begin{aligned} \oint_{C_2} F(s)e^{st} ds &= \int_L F(s)e^{st} ds + \int_{C_{R_2}} F(s)e^{st} ds \\ &= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st} ds + \int_{C_{R_2}} F(s)e^{st} ds \\ &= -2\pi j \sum_{\text{poles in } C_2} \text{Res}[F(s)e^{st}] \end{aligned}$$

Note that the residues are at the poles inside C_2 and the minus sign is due to the fact that we are dealing with a clockwise contour. Again, using Jordan's lemma, we have that for $t < 0$, $\int_{C_{R_2}} F(s)e^{st} ds \rightarrow 0$ as $R \rightarrow \infty$. So, for $t < 0$,

$$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = - \sum_{\text{poles right of } \sigma} \text{Res}[F(s)e^{st}]$$

But we know that this must be zero, since $f(t) = 0$, $t < 0$. Hence σ must be chosen such that C_2 does not contain any poles of $F(s)e^{st}$ (as $R \rightarrow \infty$), and thus C_1 must contain *all* poles of $F(s)e^{st}$. This is the answer to question (1). Note, finally, that since e^{st} is analytic everywhere (i.e. has no poles), the poles of $F(s)e^{st}$ are the same as the poles of $F(s)$. This answers question (2) and we have

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = \sum_{\text{all poles of } F(s)} \text{Res}[F(s)e^{st}]$$

Example 1

Find the inverse transform of $F(s) = \frac{1}{s+a}$.

The function $\frac{e^{st}}{s+a}$ has a simple pole at $s = -a$. Hence

$$\text{Res}_{s=-a}[F(s)e^{st}] = \lim_{s \rightarrow -a} \left[(s+a) \frac{e^{st}}{s+a} \right] = e^{-at}$$

and so,

$$\begin{aligned} L^{-1} \left[\frac{1}{s+a} \right] &= e^{-at}, \quad t \geq 0 \\ &= 0, \quad t < 0 \end{aligned}$$

Example 2

Find the inverse transform of $F(s) = \frac{1}{(s+a)^2}$.

In this case the function $\frac{e^{st}}{(s+a)^2}$ has a pole of order 2 at $s = -a$. Remember that if a function $f(z)$ has a pole of order n , then its residue at this pole is given by

$$c_1 = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]$$

In our case the required residue is:

$$\frac{1}{(2-1)!} \lim_{s \rightarrow -a} \frac{d}{ds} \left[(s+a)^2 \frac{e^{st}}{(s+a)^2} \right] = \lim_{s \rightarrow -a} [te^{st}] = te^{-at}$$

and so,

$$\begin{aligned} L^{-1} \left[\frac{1}{(s+a)^2} \right] &= te^{-at}, \quad t \geq 0 \\ &= 0, \quad t < 0 \end{aligned}$$

Example 3

Find the inverse transform of $F(s) = \frac{1}{(s+a)^2(s+b)}$.

In this case the function $\frac{e^{st}}{(s+a)^2(s+b)}$ has a pole of order 2 at $s = -a$ and a simple pole at $s = -b$. The residue at $s = -b$ is

$$\lim_{s \rightarrow -b} \left[\frac{e^{st}}{(s+a)^2} \right] = \frac{e^{-bt}}{(a-b)^2}$$

The residue at $s = -a$ is

$$\frac{1}{(2-1)!} \lim_{s \rightarrow -a} \frac{d}{ds} \left[\frac{e^{st}}{s+b} \right] = \lim_{s \rightarrow -a} \left[\frac{te^{st}}{s+b} - \frac{e^{st}}{(s+b)^2} \right] = \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}$$

and so

$$L^{-1} \left[\frac{1}{(s+a)^2(s+b)} \right] = \frac{e^{-bt}}{(a-b)^2} + \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}, \quad t \geq 0$$

Appendix: Jordan's Lemma

In the theory for inverting Laplace transforms using residue theory, we need the following result:

$$\lim_{R \rightarrow \infty} \int_{C_R} F(s)e^{st} ds = 0 \quad (1)$$

where $t > 0$ and C_R is the semicircular contour C_{R_1} shown in Figure 1a. Points on this contour are given by

$$s = \sigma + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

Looking at the table of standard transforms it can be seen that most satisfy the conditions $F(s) \rightarrow 0$ as $|s| \rightarrow \infty$ (e.g. $\frac{1}{s}$, $\frac{1}{s+a}$, etc). Therefore on C_R as $R \rightarrow \infty$, $F(s) \rightarrow 0$. This means that for any $\epsilon > 0$, an R can be found such that $|F(s)| = |F(\sigma + Re^{j\theta})| < \epsilon$. For this R we have

$$\left| \int_{C_R} F(s)e^{st} ds \right| \leq \epsilon \int_{C_R} |e^{st}| ds$$

On C_R ,

$$|e^{st}| = |e^{\sigma + Re^{j\theta}}| = |e^{(\sigma + R \cos \theta + jR \sin \theta)t}| = e^{(\sigma + R \cos \theta)t}$$

and $ds = jRe^{j\theta} d\theta$. Therefore,

$$\epsilon \int_{C_R} |e^{st}| ds = \epsilon Re^{\sigma t} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{Rt \cos \theta} d\theta = 2\epsilon Re^{\sigma t} \int_0^{\frac{\pi}{2}} e^{-Rt \sin \theta} d\theta$$

Plotting graphs of $y = \sin \theta$ and the straight line $y = \frac{2}{\pi}\theta$, we see that

$$\sin \theta \geq \frac{2}{\pi}\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

Hence the integral is less than

$$2R\epsilon e^{\sigma t} \int_0^{\frac{\pi}{2}} e^{-\frac{2Rt}{\pi}\theta} d\theta = \frac{\epsilon\pi e^{\sigma t}}{t} (1 - e^{-Rt})$$

For any $t > 0$ this last quantity goes to zero as $R \rightarrow \infty$ (because ϵ also goes to zero), and we have therefore proved (1).