**Universidade Federal do Rio de Janeiro**
COPPE – Programa de Engenharia Química

# COQ 790 – ANÁLISE DE SISTEMAS DA ENGENHARIA QUÍMICA

## AULA 12:

*Representação em Espaço de Estados*

2014/1

# Representação em Espaço de Estados

**Estado:** O estado de um sistema no tempo $t_0$ é o conjunto de informações em $t_0$ que, junto com a entrada u(t), $t \in [t_0, \infty)$, determina univocamente o comportamento do sistema para $t \geq t_0$.

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A escolha do estado não é **única!**

não linear

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t),\ \underline{u}(t),\ t]$$
$$\underline{y}(t) = \underline{g}[\underline{x}(t), \underline{u}(t), t]$$

linear variante no tempo

$$\dot{\underline{x}}(t) = \underline{\underline{A}}(t)\ \underline{x}(t) + \underline{\underline{B}}(t)\ \underline{u}(t)$$
$$\underline{y}(t) = \underline{\underline{C}}(t)\ \underline{x}(t) + \underline{\underline{D}}(t)\ \underline{u}(t)$$

LTI

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\ \underline{x}(t) + \underline{\underline{B}}\ \underline{u}(t)$$
$$\underline{y}(t) = \underline{\underline{C}}\ \underline{x}(t) + \underline{\underline{D}}\ \underline{u}(t)$$

linear homogêneo com coeficientes variáveis

$$\dot{\underline{x}}(t) = \frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}(t)\ \underline{x}(t)$$

# Matriz de Transição de Estados $\underline{\underline{\Phi}}(t;\ t_o) \in \Re^{n \times n}$

$$\left.\begin{aligned} \frac{d\underline{\underline{\Phi}}(t;\ t_0)}{dt} &= \underline{\underline{A}}(t)\ \underline{\underline{\Phi}}(t;\ t_0) \\ \underline{\underline{\Phi}}(t_0;\ t_0) &= \underline{\underline{I}} \end{aligned}\right\} \Longrightarrow \underline{x}(t) = \underline{\underline{\Phi}}(t;\ t_0)\,\underline{x}(t_0)$$

$\underline{\underline{\Phi}}(t;\ t_o)$ descreve a transição do estado de $t_0 \rightarrow t$

Integrando sucessivamente a equação homogênea:

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}(t)\ \underline{x}(t) \Longrightarrow \underline{x}(t) = \underline{x}(t_o) + \int_{t_0}^{t} \underline{\underline{A}}(\tau)\underline{x}(\tau)\ d\tau$$

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}(t)\left[\underline{x}(t_0) + \int_{t_0}^{t} \underline{\underline{A}}(\tau)\underline{x}(\tau)d\tau\right] \Longrightarrow \underline{x}(t) = \underline{x}(t_0) + \int_{t_0}^{t} \underline{\underline{A}}(\tau_1)\left[\underline{x}(t_0) + \int_{t_0}^{t} \underline{\underline{A}}(\tau_2)\underline{x}(\tau_2)d\tau_2\right]d\tau_1$$

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Após infinitas integrações chega-se a:

$$\underline{x}(t)=\underline{\underline{\Phi}}(t;t_0)\underline{x}(t_0)=\left[\underline{\underline{I}}+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)d\tau_1+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)\int_{t_0}^{\tau_1}\underline{\underline{A}}(\tau_2)d\tau_2 d\tau_1+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)\int_{t_0}^{\tau_1}\underline{\underline{A}}(\tau_2)\int_{t_0}^{\tau_2}\underline{\underline{A}}(\tau_3)d\tau_3 d\tau_2 d\tau_1+\ldots\right]\underline{x}(t_0)$$

$$\underline{\underline{\Phi}}(t;t_0)=\left[\underline{\underline{I}}+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)d\tau_1+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)\int_{t_0}^{\tau_1}\underline{\underline{A}}(\tau_2)d\tau_2 d\tau_1+\int_{t_0}^{t}\underline{\underline{A}}(\tau_1)\int_{t_0}^{\tau_1}\underline{\underline{A}}(\tau_2)\int_{t_0}^{\tau_2}\underline{\underline{A}}(\tau_3)d\tau_3 d\tau_2 d\tau_1+\ldots\right]$$

**matrizante:** série convergente quando os elementos os elementos de $\underline{\underline{A}}$ são limitados em ($t_0$, t)

Para o caso linear homogêneo com coeficientes constantes: (LTI: $t_0$ = 0)

$$\underline{\underline{\Phi}}(t;0)=\underline{\underline{I}}+\underline{\underline{A}}\,t+\underline{\underline{A}}^2\frac{t^2}{2!}+\underline{\underline{A}}^3\frac{t^3}{3!}+\ldots=\sum_{i=0}^{\infty}\underline{\underline{A}}^i\frac{t^i}{i!}=e^{\underline{\underline{A}}\,t}$$

função **matriz exponencial**

$$\left.\begin{aligned} \frac{d\underline{x}(t)}{dt} &= \underline{\underline{A}}\ \underline{x}(t) \\ \underline{x}(0) &= \underline{x}_0 \end{aligned}\right\} \Longrightarrow \underline{x}(t) = \underline{\underline{\Phi}}(t;\ 0) = e^{\underline{\underline{A}}\ t}\,\underline{x}_0$$

$$\left.\begin{aligned} \frac{d\underline{x}(t)}{dt} &= \underline{\underline{A}}\ \underline{x}(t) \\ \underline{x}(t_0) &= \underline{x}_0 \end{aligned}\right\} \Longrightarrow \underline{x}(t) = \underline{\underline{\Phi}}(t;\ t_0) = e^{\underline{\underline{A}}\ (t-t_0)}\,\underline{x}_0$$

LTI: $\underline{\underline{\Phi}}(t;\ t_0) = \underline{\underline{\Phi}}(t - t_0) = e^{\underline{\underline{A}}(t-t_0)}$

Teorema de Sylvester: $$e^{\underline{\underline{A}}t} = \sum_{i=1}^{n} e^{\lambda_i t} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{\underline{\underline{A}} - \lambda_j \underline{\underline{I}}}{\lambda_i - \lambda_j}$$

Para valores característicos iguais aplica-se a regra de L'Hopital para resolver a indeterminação.

Transformada de Laplace:

$$\left.\begin{array}{l}\dfrac{d\underline{x}(t)}{dt} = \underline{\underline{A}}\ \underline{x}(t) \\ \\ \underline{x}(0) = \underline{x}_0\end{array}\right\} \Longrightarrow \begin{array}{l}\left[s\underline{\underline{I}} - \underline{\underline{A}}\right]\underline{X}(s) = \underline{x}_0 \\ \\ \underline{x}(t) = L^{-1}\left[\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right]\underline{x}_0\end{array}$$

$$\underline{\underline{\Phi}}(t;\ 0) = \underline{\underline{\Phi}}(t) = e^{\underline{\underline{A}}t} = L^{-1}\left[\left(s\underline{\underline{I}} - \underline{\underline{A}}\right)^{-1}\right]$$

Decomposição matricial:

$$\underline{\underline{A}} = \underline{\underline{P}}\ \underline{\underline{\Lambda}}\ \underline{\underline{P}}^{-1}$$

$$e^{\underline{\underline{A}}t} = \underline{\underline{P}}\ e^{\underline{\underline{\Lambda}}t}\,\underline{\underline{P}}^{-1}$$

Para o caso linear não homogêneo com coeficientes constantes:

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}\;\underline{x}(t) + \underline{\underline{B}}\;\underline{u}(t)$$

Introduzindo a mudança de variável: $\underline{x}(t) = \underline{\underline{\Phi}}(t-t_0)\underline{z}(t) \longrightarrow \underline{z}(t) = \underline{\underline{\Phi}}^{-1}(t-t_0)\underline{x}(t)$

$$\frac{d\underline{x}(t)}{dt} = \frac{d\underline{\underline{\Phi}}}{dt}\underline{z}(t) + \underline{\underline{\Phi}}\frac{d\underline{z}(t)}{dt} = (\underline{\underline{A}}\;\underline{\underline{\Phi}})\underline{z}(t) + \underline{\underline{\Phi}}\;\frac{d\underline{z}}{dt} = \underline{\underline{A}}\;\underline{x}(t) + \underline{\underline{\Phi}}\;\frac{d\underline{z}}{dt}$$

$$\underline{\underline{\Phi}}\frac{d\underline{z}}{dt} = \frac{d\underline{x}}{dt} - \underline{\underline{A}}\;\underline{x} = \underline{\underline{B}}\;\underline{u}$$

$$\frac{d\underline{z}}{dt} = \underline{\underline{\Phi}}^{-1}\underline{\underline{B}}\;\underline{u} \quad \Rightarrow \quad \underline{z}(t) = \underline{z}(t_0) + \int_{t_o}^{t} \underline{\underline{\Phi}}^{-1}(\tau - t_0)\underline{\underline{B}}\;\underline{u}(\tau)\;d\tau$$

$$\underline{x}(t) = \underline{\underline{\Phi}}(t-t_0)\underline{z}(t_0) + \underline{\underline{\Phi}}(t-t_0)\int_{t_0}^{t} \underline{\underline{\Phi}}^{-1}(\tau - t_0)\underline{\underline{B}}\;\underline{u}(\tau)\;d\tau$$

$$\underline{\underline{\Phi}}(t-t_0)=e^{\underline{\underline{A}}(t-t_0)} \longrightarrow \underline{\underline{\Phi}}^{-1}(\tau-t_0)=e^{-\underline{\underline{A}}(\tau-t_0)}=e^{\underline{\underline{A}}(t_0-\tau)}=\underline{\underline{\Phi}}(t_0-\tau)$$

$$\longrightarrow \underline{\underline{\Phi}}(t-t_0)\cdot\underline{\underline{\Phi}}(t_0-\tau)=e^{\underline{\underline{A}}(t-t_0)}\cdot e^{\underline{\underline{A}}(t_0-\tau)}=e^{\underline{\underline{A}}(t-\tau)}=\underline{\underline{\Phi}}(t-\tau)$$

$$\underline{z}(t)=\underline{\underline{\Phi}}^{-1}(t-t_0)\underline{x}(t) \longrightarrow \underline{z}(t_0)=\underline{\underline{\Phi}}^{-1}(t_0-t_0)\underline{x}(t_0)=\underline{x}(t_0)$$

$$\underline{x}(t)=\underline{\underline{\Phi}}(t-t_0)\underline{x}(t_0)+\int_{t_0}^{t}\underline{\underline{\Phi}}(t-\tau)\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$$

$$\underline{x}(t)=\underbrace{e^{\underline{\underline{A}}(t-t_0)}\underline{x}(t_0)}_{(1)}+\underbrace{\int_{t_0}^{t}e^{\underline{\underline{A}}(t-\tau)}\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau}_{(2)}$$

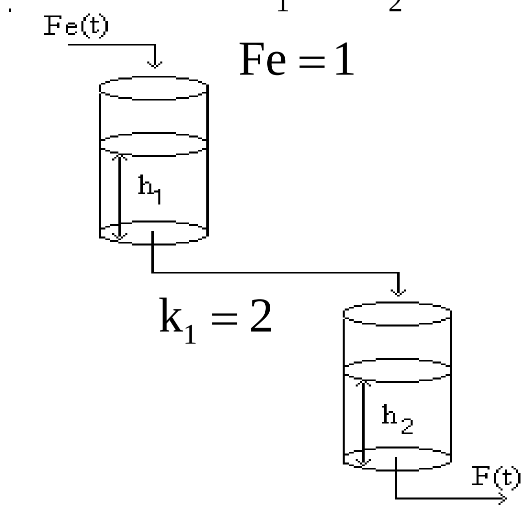
(1) resposta à entrada nula

(2) resposta ao estado nulo

Exemplo:

$A_1 = A_2 = 1$

$Fe = 1$




$k_1 = 2$

$k_2 = \sqrt{2}$

$$A_1 \frac{dh_1(t)}{dt} = Fe(t) - k_1\sqrt{h_1(t)}$$

$$A_2 \frac{dh_2(t)}{dt} = k_1\sqrt{h_1(t)} - k_2\sqrt{h_2(t)}$$

Linearizando em torno do estado estacionário:

$$x_1(t) = h_1(t) - \bar{h}_1$$

$$x_2(t) = h_2(t) - \bar{h}_2$$

$$u(t) = F_e(t) - \bar{F}_e$$

$$\frac{dx_1(t)}{dt} = a_{11} \cdot x_1(t) + b_1 \cdot u(t)$$

$$\frac{dx_2(t)}{dt} = a_{21} \cdot x_1(t) + a_{22} \cdot x_2(t)$$

$$a_{11} = \frac{-k_1}{2\sqrt{\bar{h}_1}} \cdot \frac{1}{A_1} \qquad b_1 = \frac{1}{A_1}$$

$$a_{21} = \frac{k_1}{2\sqrt{\bar{h}_1}} \cdot \frac{1}{A_2} \qquad a_{22} = \frac{-k_2}{2\sqrt{\bar{h}_2}} \cdot \frac{1}{A_2}$$

Se a variável medida na saída é o nível do segundo tanque:

$$y(t) = h_2(t) - \bar{h}_2$$

$$\underline{\dot{x}}(t) = \underline{\underline{A}} \cdot \underline{x}(t) + \underline{b} \cdot u(t) \qquad \underline{\underline{A}} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y(t) = \underline{c}^T \cdot \underline{x}(t)$$

Estado estacionário:

$$\overline{Fe} - k_1\sqrt{\overline{h}_1} = 0 \qquad \overline{h}_1 = 1/4$$

$$k_1\sqrt{\overline{h}_1} - k_2\sqrt{\overline{h}_2} = 0 \qquad \overline{h}_2 = 1/2$$

$$\underline{\underline{A}} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Utilizando o teorema de Sylvester:

$$\left|\underline{\underline{A}} - \lambda \cdot \underline{\underline{I}}\right| = \begin{vmatrix} -2-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 3 \cdot \lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases}$$

$$e^{\underline{\underline{A}}(t-t_0)} = e^{-(t-t_0)} \frac{\begin{pmatrix} -2 & 0 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}}{-1-(-2)} + e^{-2(t-t_0)} \frac{\begin{pmatrix} -2 & 0 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}{-2-(-1)} = \begin{pmatrix} e^{-2(t-t_0)} & 0 \\ 2e^{-(t-t_0)} - 2e^{-2(t-t_0)} & e^{-(t-t_0)} \end{pmatrix}$$

$$\underline{x}(t)=\begin{pmatrix} e^{-2(t-t_0)}\cdot x_1(t_0) \\ e^{-(t-t_0)}[2\cdot x_1(t_0)+x_2(t_0)]-2\cdot e^{-2(t-t_0)}\cdot x_1(t_0)\end{pmatrix}+\begin{pmatrix} \int_{t_0}^{t} e^{-2(t-\tau)}\cdot u(\tau)d\tau \\ \int_{t_0}^{t}\left[2\cdot e^{-(t-\tau)}-2\cdot e^{-2(t-\tau)}\right]\cdot u(\tau)d\tau \end{pmatrix}$$

$$y(t)=e^{-(t-t_0)}[2\cdot x_1(t_0)+x_2(t_0)]-2\cdot e^{-2(t-t_0)}\cdot x_1(t_0)+\int_{t_0}^{t}2\cdot e^{-(t-\tau)}\cdot u(\tau)d\tau-\int_{t_0}^{t}2\cdot e^{-2(t-\tau)}\cdot u(\tau)d\tau$$

Utilizando o transformada de Laplace:

$$\left(s\underline{\underline{I}}-\underline{\underline{A}}\right)\underline{X}(s)=\underline{x}_0+\underline{b}\,U(s) \longrightarrow \underline{x}(t)=L^{-1}\left[\left(s\underline{\underline{I}}-\underline{\underline{A}}\right)^{-1}\right]\underline{x}_0+L^{-1}\left[\left(s\underline{\underline{I}}-\underline{\underline{A}}\right)^{-1}\underline{b}\,U(s)\right]$$

$$e^{\underline{\underline{A}}t}=L^{-1}\left[\left(s\underline{\underline{I}}-\underline{\underline{A}}\right)^{-1}\right]=L^{-1}\left[\begin{bmatrix} s+2 & 0 \\ -2 & s+1\end{bmatrix}^{-1}\right]=L^{-1}\begin{bmatrix} \frac{1}{(s+2)} & 0 \\ \frac{2}{(s-2)(s-1)} & \frac{1}{(s+1)}\end{bmatrix}$$

$$e^{\underline{\underline{A}}t}=\begin{pmatrix} e^{-2t} & 0 \\ 2e^{-t}-2e^{-2t} & e^{-t}\end{pmatrix}$$

Utilizando o decomposição matricial:

$$\underline{\underline{A}} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e^{\underline{\underline{A}}t} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ 2e^{-t} - 2e^{-2t} & e^{-t} \end{pmatrix}$$

# Representação em Espaço de Estados Discretos

| não linear | linear variante no tempo | LTI |
| --- | --- | --- |
| $\underline{x}(k+1) = \underline{f}\left[\underline{x}(k), \underline{u}(k), k\right]$ | $\underline{x}(k+1) = \underline{\underline{A}}(k)\underline{x}(k) + \underline{\underline{B}}(k)\underline{u}(k)$ | $\underline{x}(k+1) = \underline{\underline{A}}\,\underline{x}(k) + \underline{\underline{B}}\,\underline{u}(k)$ |
| $\underline{y}(k) = \underline{g}\left[\underline{x}(k), \underline{u}(k), k\right]$ | $\underline{y}(k) = \underline{\underline{C}}(k)\underline{x}(k) + \underline{\underline{D}}(k)\underline{u}(k)$ | $\underline{y}(k) = \underline{\underline{C}}\,\underline{x}(k) + \underline{\underline{D}}\,\underline{u}(k)$ |

linear homogêneo com coeficientes constantes

$$\underline{x}(k+1) = \underline{\underline{A}}\,\underline{x}(k) \qquad , \qquad \underline{x}(0) = \underline{x}_0$$

$$\underline{x}(1) = \underline{\underline{A}}\,\underline{x}_0$$

$$\underline{x}(2) = \underline{\underline{A}}\,\underline{x}(1) = \underline{\underline{A}}\,\underline{\underline{A}}\,\underline{x}_0 = \underline{\underline{A}}^2\,\underline{x}_0$$

$$\vdots$$

$$\underline{x}(k) = \boxed{\underline{\underline{A}}^k}\,\underline{x}_0$$

$\underline{\underline{A}}^k$ → Matriz de transição de estados discretos

Teorema de Sylvester: $$\underline{\underline{A}}^k = \sum_{i=1}^{n} \lambda_i^k \prod_{\substack{j=1 \\ j \neq i}}^{n} \left( \frac{\underline{\underline{A}} - \lambda_j \underline{\underline{I}}}{\lambda_i - \lambda_j} \right)$$

Decomposição matricial: $$\underline{\underline{A}}^k = \underline{\underline{P}}\,\underline{\underline{\Lambda}}^k\,\underline{\underline{P}}^{-1}$$

$$\lim_{k \to \infty} \underline{\underline{\Lambda}}^k = \underline{\underline{0}}$$ que ocorre quando $|\lambda_i| < 1$

Para o caso linear não homogêneo com coeficientes constantes:

$$\underline{x}(k+1) = \underline{\underline{A}}\,\underline{x}(k) + \underline{\underline{B}}\,\underline{u}(k)$$

$$\underline{x}(1) = \underline{\underline{A}}\,\underline{x}_0 + \underline{\underline{B}}\,\underline{u}(0)$$

$$\underline{x}(2) = \underline{\underline{A}}\,\underline{x}(1) + \underline{\underline{B}}\,\underline{u}(1) = \underline{\underline{A}}\left[\underline{\underline{A}}\,\underline{x}_0 + \underline{\underline{B}}\,\underline{u}(0)\right] + \underline{\underline{B}}\,\underline{u}(1) = \underline{\underline{A}}^2\,\underline{x}_0 + \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{u}(0) + \underline{\underline{B}}\,\underline{u}(1)$$

$$\vdots$$

$$\underline{x}(k) = \underline{\underline{A}}^k\,\underline{x}_0 + \sum_{i=0}^{k-1} \underline{\underline{A}}^{k-i-1}\,\underline{\underline{B}}\,\underline{u}(i)$$

$$\dot{\underline{x}}(t)=\underline{\underline{A}}\,\underline{x}(t)+\underline{\underline{B}}\,\underline{u}(t) \quad \Longrightarrow \quad \underline{x}(t)=e^{\underline{\underline{A}}(t-t_0)}\underline{x}(t_0)+\int_{t_0}^{t} e^{\underline{\underline{A}}(t-\tau)}\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$$

**Solução exata**,

considerando $\underline{u}(t)=\underline{u}(t_k)$ no intervalo $k\,T_a < t \leq (k+1)\,T_a$ (dado amostrado)

$$\underline{x}(t)=\underbrace{e^{\underline{\underline{A}}(t-t_k)}}_{\underline{\underline{\Phi}}(t-t_k)}\underline{x}(t_k)+\underbrace{\left[\int_{t_k}^{t} e^{\underline{\underline{A}}(t-\tau)}\underline{\underline{B}}\,d\tau\right]}_{\underline{\underline{\Gamma}}(t-t_k)}\underline{u}(t_k)$$

Para t = (k+1) $T_a$

$$\underline{x}(k+1)=\underline{\underline{\Phi}}(T_a)\,\underline{x}(k)+\underline{\underline{\Gamma}}(T_a)\,u(k)$$

$$\underline{\underline{\Phi}}(T_a)=e^{\underline{\underline{A}}T_a} \qquad \underline{\underline{\Gamma}}(T_a)=\int_0^{T_a} e^{\underline{\underline{A}}\tau}\,\underline{\underline{B}}\,d\tau$$

$$\underline{x}(k+1)=\underline{\underline{A}}_d\,\underline{x}(k)+\underline{\underline{B}}_d\,\underline{u}(k) \quad \Longrightarrow \quad \underline{x}(k)=\underline{\underline{A}}_d^{\,k}\,\underline{x}_0+\sum_{i=0}^{k-1}\underline{\underline{A}}_d^{\,k-i-1}\,\underline{\underline{B}}_d\,\underline{u}(i)$$

$$\underline{x}(k+1)=\underline{\underline{\Phi}}(T_a)\,\underline{x}(k)+\underline{\underline{\Gamma}}(T_a)\,u(k)$$

$$\underline{x}(k)=\underline{\underline{\Phi}}(T_a)^k\,\underline{x}_0+\sum_{i=0}^{k-1}\underline{\underline{\Phi}}(T_a)^{k-i-1}\,\underline{\underline{\Gamma}}(T_a)\,\underline{u}(i)$$

$$\underline{\underline{\Phi}}(T_a)^k=\left(e^{\underline{\underline{A}}T_a}\right)^k=e^{\underline{\underline{A}}\,k\,T_a}=\underline{\underline{\Phi}}(kT_a)$$

$$\underline{x}(k)=\underline{\underline{\Phi}}(kT_a)\,\underline{x}_0+\sum_{i=0}^{k-1}\underline{\underline{\Phi}}\left[(k-i-1)T_a\right]\,\underline{\underline{\Gamma}}(T_a)\,\underline{u}(i)$$

**Conversão contínuo → discreto:**

$$\underline{\underline{A}}_d=\underline{\underline{\Phi}}(T_a) \qquad \underline{\underline{B}}_d=\underline{\underline{\Gamma}}(T_a)$$

$$\underline{\dot{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

**Solução aproximada**

Aplicando o método de **Euler explícito**: $\underline{\dot{x}}(t_{k+1}) \approx \frac{\underline{x}(t_{k+1}) - \underline{x}(t_k)}{T_a} = \underline{\underline{A}}\,\underline{x}(t_k) + \underline{\underline{B}}\,\underline{u}(t_k)$

$$\underline{x}(t_{k+1}) = \left(\underline{\underline{I}} + \underline{\underline{A}}T_a\right)\underline{x}(t_k) + \underline{\underline{B}}T_a\,\underline{u}(t_k) \qquad T_a < \frac{1}{\lambda_{max\_abs}(\underline{\underline{A}})}$$

$$\underline{\underline{A}}_d = \underline{\underline{\Phi}}(T_a) \approx \underline{\underline{I}} + \underline{\underline{A}}\,T_a \qquad \underline{\underline{B}}_d = \underline{\underline{\Gamma}}(T_a) = \int_0^{T_a} e^{\underline{\underline{A}}\tau}\,\underline{\underline{B}}\,d\tau \approx \underline{\underline{B}}\,T_a$$

Aplicando o método de **Euler implícito**: $\underline{\dot{x}}(t_{k+1}) \approx \frac{\underline{x}(t_{k+1}) - \underline{x}(t_k)}{T_a} = \underline{\underline{A}}\,\underline{x}(t_{k+1}) + \underline{\underline{B}}\,\underline{u}(t_k)$

considerando $\underline{u}(t) = \underline{u}(t_k)$ no intervalo $k\,T_a < t \le (k+1)\,T_a$ (dado amostrado)

$$\underline{x}(t_{k+1}) = \left(\underline{\underline{I}} - \underline{\underline{A}}T_a\right)^{-1}\underline{x}(t_k) + \left(\underline{\underline{I}} - \underline{\underline{A}}T_a\right)^{-1}\underline{\underline{B}}T_a\,\underline{u}(t_k)$$

$$\underline{\underline{A}}_d = \underline{\underline{\Phi}}(T_a) \approx \left(\underline{\underline{I}} - \underline{\underline{A}}\,T_a\right)^{-1} \qquad \underline{\underline{B}}_d = \underline{\underline{\Gamma}}(T_a) \approx \left(\underline{\underline{I}} - \underline{\underline{A}}\,T_a\right)^{-1}\underline{\underline{B}}\,T_a$$

# Diagonalização

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{\underline{B}}\,\underline{u}(t)$$

$$\underline{\underline{A}} = \underline{\underline{P}}\;\underline{\underline{\Lambda}}\;\underline{\underline{P}}^{-1}$$

$$e^{\underline{\underline{A}}\,t} = \underline{\underline{P}}\;e^{\underline{\underline{\Lambda}}\,t}\,\underline{\underline{P}}^{-1}$$

Mudança de variável: $\underline{x}^{*} = \underline{\underline{P}}^{-1}\underline{x}$

$$\underline{\underline{P}}^{-1}\frac{d\underline{x}}{dt} = \frac{d\left(\underline{\underline{P}}^{-1}\underline{x}\right)}{dt} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\;\underline{\underline{P}}\;\underline{\underline{P}}^{-1}\underline{x} \Rightarrow \frac{d\underline{x}^{*}}{dt} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\;\underline{\underline{P}}\;\underline{x}^{*} = \underline{\underline{\Lambda}}\;\underline{x}^{*}$$

$$\left.\begin{aligned} \frac{dx_1^{*}}{dt} &= \lambda_1 x_1^{*} \\ \frac{dx_2^{*}}{dt} &= \lambda_2 x_2^{*} \\ &\vdots \\ \frac{dx_n^{*}}{dt} &= \lambda_n x_n^{*} \end{aligned}\right\} \Rightarrow x_i^{*}(t) = e^{\lambda_i (t-t_0)} x_i^{*}\left(t_0\right)$$

$$\underline{x}^*(t) = \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{bmatrix} \underline{x}^*(t_0) = e^{\underline{\underline{\Lambda}}(t-t_0)} \underline{x}^*(t_0)$$

$$\underline{x}(t) = \underline{\underline{P}}\ \underline{x}^*(t) \quad \Longrightarrow \quad \underline{x}(t) = \underline{\underline{P}}\ e^{\underline{\underline{\Lambda}}(t-t_0)} \underline{\underline{P}}^{-1} \underline{x}(t_0) = e^{\underline{\underline{A}}(t-t_0)} \underline{x}(t_0)$$

$$\underline{\underline{\Phi}}(t-t_0) = e^{\underline{\underline{A}}(t-t_0)} = \underline{\underline{P}}\ e^{\underline{\underline{\Lambda}}(t-t_0)} \underline{\underline{P}}^{-1}$$

Valores e vetores característicos da matriz A:

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{\Lambda}} \Rightarrow \begin{cases} \underline{\underline{A}}\ \underline{\underline{P}} = \underline{\underline{P}}\ \underline{\underline{\Lambda}} \\ \underline{\underline{P}}^{-1}\underline{\underline{A}} = \underline{\underline{\Lambda}}\ \underline{\underline{P}}^{-1} \end{cases}$$

à direita:

$$\underline{\underline{P}} = \begin{bmatrix} \underline{v}_{d1} & \underline{v}_{d2} & \cdots & \underline{v}_{dn} \end{bmatrix}$$

$$\underline{\underline{A}}\ \underline{v}_{dj} = \lambda_j\ \underline{v}_{dj}$$

à esquerda:

$$\underline{\underline{P}}^{-1} = \begin{bmatrix} \underline{v}_{e1}^T \\ \underline{v}_{e2}^T \\ \vdots \\ \underline{v}_{en}^T \end{bmatrix}$$

$$\underline{v}_{ej}^T\ \underline{\underline{A}} = \lambda_j\ \underline{v}_{ej}^T$$

$$\underline{x}(t) = \underline{\underline{P}}\, e^{\underline{\underline{\Lambda}}(t-t_0)}\underline{\underline{P}}^{-1}\underline{x}(t_0) + \int_{t_0}^{t} \underline{\underline{P}}\, e^{\underline{\underline{\Lambda}}(t-\tau)}\underline{\underline{P}}^{-1}\underline{\underline{B}}\,\underline{u}(\tau)d\tau$$

$$\underline{x}(t) = \begin{bmatrix} \underline{v}_{d1} & \ldots & \underline{v}_{dn} \end{bmatrix} \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2(t-t_0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n(t-t_0)} \end{pmatrix} \begin{bmatrix} \underline{v}_{e1}^T \\ \underline{v}_{e2}^T \\ \vdots \\ \underline{v}_{en}^T \end{bmatrix} \underline{x}(t_0) + \int_{t_o}^{t} \begin{bmatrix} \underline{v}_{d1} & \ldots & \underline{v}_{dn} \end{bmatrix} \begin{pmatrix} e^{\lambda_1(t-\tau)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n(t-\tau)} \end{pmatrix} \cdot \begin{bmatrix} \underline{v}_{e1}^T \\ \underline{v}_{e2}^T \\ \vdots \\ \underline{v}_{en}^T \end{bmatrix} \underline{\underline{B}}\ \underline{u}(\tau)\ d\tau$$

$$\underline{x}(t)=\left(\underline{v}_{d1}e^{\lambda_1(t-t_0)}\ldots\underline{v}_{dn}e^{\lambda_n(t-t_0)}\right)\begin{pmatrix}\underline{v}_{e1}^T\,\underline{x}(t_0)\\ \\ \underline{v}_{en}^T\,\underline{x}(t_0)\end{pmatrix}+\int_{t_o}^{t}\left(\underline{v}_{d1}e^{\lambda_1(t-\tau)}\ldots\underline{v}_{dn}e^{\lambda_n(t-\tau)}\right)\begin{pmatrix}\underline{v}_{e1}^T\,\underline{\underline{B}}\,\underline{u}(\tau)\\ \\ \underline{v}_{en}^T\,\underline{\underline{B}}\,\underline{u}(\tau)\end{pmatrix}d\tau$$

$$\underline{x}(t)=\sum_{j=1}^{n}\underline{v}_{d\,j}\;e^{\lambda_j(t-t_0)}\;\underline{v}_{ej}^T\;\underline{x}(t_0)+\int_{t_0}^{t}\sum_{j=1}^{n}\underline{v}_{d\,j}e^{\lambda_j(t-\tau)}\,\underline{v}_{ej}^T\;\underline{\underline{B}}\;\underline{u}(\tau)\;d\tau$$

$\underline{v}_{ej}^T\;\underline{x}(t_0)$ → **ativação** do modo

$e^{\lambda_j(t-t_0)}$ → **modo** da resposta

$\underline{v}_{d\,j}$ → **composição** do modo

O modo vinculado a $\lambda_j$ contribui de forma diferente para cada variável de estado:

mesma ativação do modo j para todos os estados: $\left[v_{ej}^1 x_1(t_0)+v_{ej}^2 x_2(t_0)+\ldots+v_{ej}^n x_n(t_0)\right]$

$$x_i(t)=\sum_{j=1}^{n}v_{dj}^{i}\,e^{\lambda_j(t-t_o)}\left[v_{ej}^1 x_1(t_0)+v_{ej}^2 x_2(t_0)+\ldots+v_{ej}^n x_n(t_0)\right]+\int_{t_0}^{t}\sum_{j=1}^{n}v_{dj}^{i}\,e^{\lambda_j(t-\tau)}\underline{v}_{ej}^T\;\underline{\underline{B}}\;\underline{u}(\tau)\;d\tau$$

$v_{dj}^{i}$ → contribuição do modo j para a resposta temporal do estado i

Para o sistema diagonalizado:

$$\underline{x}^*(t) = e^{\underline{\underline{\Lambda}}(t-t_o)}\underline{x}^*(t_0) = \begin{bmatrix} e^{\lambda_1(t-t_0)}x_1^*(t_0) \\ e^{\lambda_2(t-t_0)}x_2^*(t_0) \\ \vdots \\ e^{\lambda_n(t-t_0)}x_n^*(t_0) \end{bmatrix}$$

cada modo contribui com peso unitário para uma única variável de estado e sua ativação é dada pela condição inicial da respectiva variável de estado.

# Controlabilidade

$$\dot{\underline{x}}(t)=\underline{\underline{A}}\,\underline{x}(t)+\underline{\underline{B}}\,\underline{u}(t) \Longrightarrow \underline{x}(t)=\sum_{j=1}^{n}\underline{v}_{dj}\,e^{\lambda_j(t-t_0)}\,\underline{v}_{ej}^{T}\,\underline{x}(t_0)+\int_{t_0}^{t}\sum_{j=1}^{n}\underline{v}_{dj}\,e^{\lambda_j(t-\tau)}\,\underline{v}_{ej}^{T}\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau$$

Como $\underline{u}(t)$ só afeta o segundo termo, vamos considerar $\underline{x}(t_0) = 0$:

$$\underline{x}(t)=\int_{t_0}^{t}\sum_{j=1}^{n}\underline{v}_{dj}e^{\lambda_j(t-\tau)}\underline{v}_{ej}^{T}\,\underline{\underline{B}}\,\underline{u}(\tau)\,d\tau=\int_{t_0}^{t}\sum_{j=1}^{n}\underline{v}_{dj}e^{\lambda_j(t-\tau)}\underline{v}_{ej}^{T}\left[\underline{b}_1\ \underline{b}_2\cdots\underline{b}_n\right]\underline{u}(\tau)\,d\tau$$

Se $\underline{v}_{ej}^{T}\left(\underline{b}_1\ \underline{b}_2\ \cdots\ \underline{b}_m\right)=\underline{v}_{ej}^{T}\,\underline{\underline{B}}=\underline{0}^{T}$

então $\underline{u}(t)$ não influencia o modo j: este **modo não é controlável**.

Analisando o sistema diagonalizado:

estado $x_j^*$ não é controlável

$$\frac{d\underline{x}^*}{dt}=\underline{\underline{\Lambda}}\,\underline{x}^*+\underline{\underline{P}}^{-1}\underline{\underline{B}}\,\underline{u}$$

$$\frac{dx_j^*}{dt}=\lambda_j\,x_j^*+\underline{v}_{ej}^{T}\,\underline{\underline{B}}\,\underline{u}=\lambda_j\,x_j^*+\underline{0}^{T}\,\underline{u}=\lambda_j\,x_j^*$$

Matriz de controlabilidade:

$$\underline{\underline{K}} \overset{\Delta}{=} \left[\underline{\underline{B}} \;\; \underline{\underline{A}}\,\underline{\underline{B}} \;\; \underline{\underline{A}}^2 \underline{\underline{B}} \cdots \underline{\underline{A}}^{n-1} \underline{\underline{B}}\right]$$

O sistema é **controlável** se $\operatorname{posto}\left(\underline{\underline{K}}\right) = n$

Ilustrando com um sistema discreto com condição inicial nula:

$$\underline{x}(k+1) = \underline{\underline{A}}\,\underline{x}(k) + \underline{\underline{B}}\,\underline{u}(k)$$

$$\underline{x}(1) = \underline{\underline{B}}\;\underline{u}(0)$$

$$\underline{x}(2) = \underline{\underline{A}}\;\underline{x}(1) + \underline{\underline{B}}\;\underline{u}(1) = \underline{\underline{A}}\;\underline{\underline{B}}\;\underline{u}(0) + \underline{\underline{B}}\;\underline{u}(1)$$

$$\underline{x}(3) = \underline{\underline{A}}\;\underline{x}(2) + \underline{\underline{B}}\;\underline{u}(2) = \underline{\underline{A}}^2\;\underline{\underline{B}}\;\underline{u}(0) + \underline{\underline{A}}\;\underline{\underline{B}}\;\underline{u}(1) + \underline{\underline{B}}\;\underline{u}(2)$$

$$\vdots$$

$$\underline{x}(n) = \sum_{i=0}^{n-1} \underline{\underline{A}}^{n-i-1}\;\underline{\underline{B}}\;\underline{u}(i)$$

Considerando um sistema de segunda ordem com uma única entrada e $\underline{x}(t_0) = 0$:

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{b}\,u(t)$$

Aplicando uma entrada $u_I$ durante um pequeno intervalo $\Delta t$:

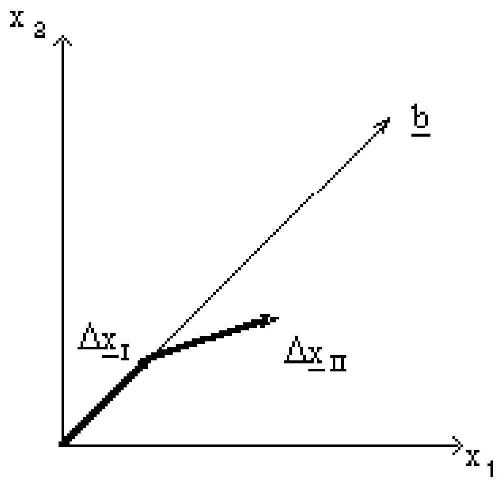
$$\Delta\underline{x}_I(t) \approx \underline{b}\,u_I\,\Delta t$$

Aplicando uma segunda entrada $u_{II}$ durante o seguinte intervalo $\Delta t$:

$$\Delta\underline{x}_{II} \approx \underline{\underline{A}}\,\Delta\underline{x}_I\Delta t + \underline{b}\,u_{II}\,\Delta t = \underline{\underline{A}}\;\underline{b}\,u_I\,(\Delta t)^2 + \underline{b}\,u_{II}\,\Delta t$$

Para poder alcançar qualquer ponto do plano $(x_1, x_2)$ é necessário que os vetores $\Delta x_I$ e $\Delta x_{II}$ não tenham a mesma direção, isto é, que sejam L.I.:

$\underline{\underline{K}} = \left[\underline{b} \quad \underline{\underline{A}}\,\underline{b}\right]$ ⟹ deve ter posto 2



# Observabilidade

$$\underline{\dot{x}}(t) = \underline{\underline{A}}(t)\ \underline{x}(t) + \underline{\underline{B}}(t)\ \underline{u}(t)$$

$$\underline{y}(t) = \underline{\underline{C}}(t)\ \underline{x}(t) + \underline{\underline{D}}(t)\ \underline{u}(t)$$

Considerando $\underline{u}(t) = 0$:

$$\underline{\dot{x}}(t) = \underline{\underline{A}}(t)\ \underline{x}(t) \qquad \underline{x}(t_0) = \underline{x}_0$$

$$\underline{y}(t) = \underline{\underline{C}}(t)\ \underline{x}(t)$$

$$\underline{y}(t) = \underline{\underline{C}}\ \underline{x}(t) = \underline{\underline{C}}\ \underline{\underline{P}}\ \underline{x}^*(t) = \underline{\underline{C}}\left[\underline{v}_{d1}\ \ \underline{v}_{d2} \cdots \underline{v}_{dn}\right]\begin{bmatrix} x_1^*(t) \\ x_2^*(t) \\ \vdots \\ x_n^*(t) \end{bmatrix} = \left[\underline{\underline{C}}\underline{v}_{d1}\ \ \underline{\underline{C}}\underline{v}_{d2} \cdots \underline{\underline{C}}\underline{v}_{dn}\right]\begin{bmatrix} e^{\lambda_1(t-t_0)}x_1^*(t_0) \\ e^{\lambda_2(t-t_0)}x_2^*(t_0) \\ \vdots \\ e^{\lambda_n(t-t_0)}x_n^*(t_0) \end{bmatrix}$$

Se $\quad \underline{\underline{C}}\,\underline{v}_{dj} = \underline{0}$

então o modo j não influencia $\underline{y}(t)$: este **modo não é observável**.

Matriz de observabilidade:

$$\underline{\underline{L}} \stackrel{\Delta}{=} \left[ \underline{\underline{C}}^T \quad \underline{\underline{A}}^T \underline{\underline{C}}^T \quad \left(\underline{\underline{A}}^2\right)^T \underline{\underline{C}}^T \quad \cdots \quad \left(\underline{\underline{A}}^{n-1}\right)^T \underline{\underline{C}}^T \right]$$

O sistema é **observável** se $\text{posto}\left(\underline{\underline{L}}\right) = n$

Ilustrando com um sistema discreto com entrada nula:

$$\underline{x}(k+1) = \underline{\underline{A}}\,\underline{x}(k) \qquad \underline{y}(k) = \underline{\underline{C}}\,\underline{x}(k)$$

$$\underline{y}(0) = \underline{\underline{C}}\;\underline{x}(0) = \underline{\underline{C}}\;\underline{x}_0$$

$$\underline{y}(1) = \underline{\underline{C}}\;\underline{x}(1) = \underline{\underline{C}}\underline{\underline{A}}\;\underline{x}_0$$

$$\underline{y}(2) = \underline{\underline{C}}\;\underline{x}(2) = \underline{\underline{C}}\underline{\underline{A}}\;\underline{x}(1) = \underline{\underline{C}}\underline{\underline{A}}^2\underline{x}_0$$

$$\vdots$$

$$\underline{y}(n-1) = \underline{\underline{C}}\underline{\underline{A}}^{n-1}\underline{x}_0$$

A definição de controlabilidade é:

“Um sistema é controlável se e somente se, para todo estado inicial, $\underline{x}(t_0)$, existe uma entrada contínua por partes, $\underline{u}(t_0, t_1]$ tal que $\underline{x}(t_1) = 0$ para algum $t_1$ finito”.

A definição de observabilidade é:

“Um sistema é observável se e somente se, para todo $t_1 \geq t_0$, a observação de $\underline{y}(t_0, t_1]$, para qualquer $\underline{u}(t_0, t_1]$ conhecido, permite calcular $\underline{x}(t_0)$”.

1) O modo é controlável é observável

$$\frac{dx_j^*}{dt} = \lambda_j\, x_j + \underline{v}_{ej}^T\, \underline{\underline{B}}\, \underline{u}$$

$$\underline{y}(t) = \begin{bmatrix} \cdots & \underline{\underline{C}}\underline{v}_{dj} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{\lambda_j(t-t_0)} x_j^*(t_0) \\ \vdots \end{bmatrix}$$

2) O modo é controlável e não observável

$$\frac{dx_j^*}{dt} = \lambda_j\, x_j + \underline{v}_{ej}^T\, \underline{\underline{B}}\, \underline{u}$$

$$\underline{y}(t) = \begin{bmatrix} \cdots & \underline{0} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{\lambda_j(t-t_0)} x_j^*(t_0) \\ \vdots \end{bmatrix}$$

3) O modo é não controlável e observável

$$\frac{dx_j^*}{dt} = \lambda_j\, x_j + \underline{0}^T\, \underline{u}$$

$$\underline{y}(t) = \begin{bmatrix} \cdots & \underline{\underline{C}}\underline{v}_{dj} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{\lambda_j(t-t_0)} x_j^*(t_0) \\ \vdots \end{bmatrix}$$

4) O modo não é nem controlável nem observável

$$\frac{dx_j^*}{dt} = \lambda_j\, x_j + \underline{0}^T\, \underline{u}$$

$$\underline{y}(t) = \begin{bmatrix} \cdots & \underline{0} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ e^{\lambda_j(t-t_0)} x_j^*(t_0) \\ \vdots \end{bmatrix}$$

Particionando o vetor de estados nas quatro categorias possíveis:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}^*_{c-o} \\ \underline{x}^*_{c-no} \\ \underline{x}^*_{nc-o} \\ \underline{x}^*_{nc-no} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\Lambda}}_{c-o} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\Lambda}}_{c-no} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\Lambda}}_{nc-o} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\Lambda}}_{nc-no} \end{bmatrix} \begin{bmatrix} \underline{x}^*_{c-o} \\ \underline{x}^*_{c-no} \\ \underline{x}^*_{nc-o} \\ \underline{x}^*_{nc-no} \end{bmatrix} + \begin{bmatrix} \underline{\underline{P}}^{-1}_{c-o} \\ \underline{\underline{P}}^{-1}_{c-no} \\ \underline{\underline{0}} \\ \underline{\underline{0}} \end{bmatrix} \underline{\underline{B}}\,\underline{u}$$

$$\underline{y} = \underline{\underline{C}}\begin{bmatrix} \underline{\underline{P}}_{c-o} & \underline{\underline{0}} & \underline{\underline{P}}_{nc-o} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \underline{x}^*_{c-o} \\ \underline{x}^*_{c-no} \\ \underline{x}^*_{nc-o} \\ \underline{x}^*_{nc-no} \end{bmatrix}$$

Único caminho possível:

$$\underline{y}(s) = \left[\underline{\underline{C}}\,\underline{\underline{P}}_{c-o}\left(s\,\underline{\underline{I}} - \underline{\underline{\Lambda}}_{c-o}\right)^{-1}\underline{\underline{P}}^{-1}_{c-o}\,\underline{\underline{B}}\right]\underline{u}(s)$$