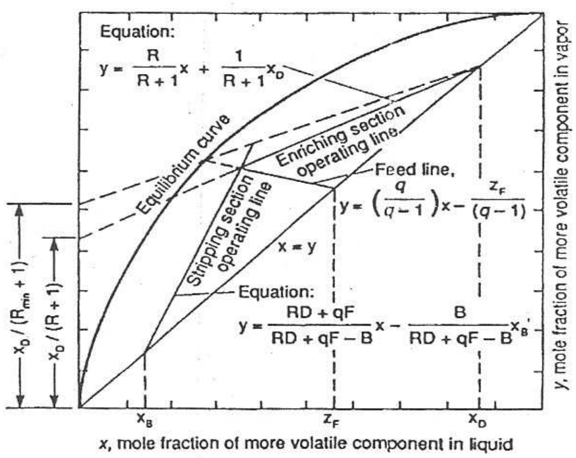
Principais Estruturas de Controle para uma Coluna de Destilação Simples

- As estruturas de controle são denominadas segundo as variáveis manipuladas usadas para o controle de das composições.
- Dada a propriedade integrativa apresentada pelos acúmulos estes deverão estar sempre em malha fechada.
- Dependendo se uma ou ambas as composições são controladas em malha fechada fala-se em "One-" oder "Two-point-control"("Dual composition control").

$$\begin{bmatrix} y_D \\ x_B \\ M_D \\ M_B \\ M_V \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) & 0 & 0 & 0 \\ g_{21}(s) & g_{22}(s) & 0 & 0 & 0 \\ -\frac{1}{s} & 0 & -\frac{1}{s} & 0 & \frac{1}{s} \\ \frac{e^{-\theta s}}{s} & \frac{\left(\frac{\partial L_N}{\partial V_N}\right)_{M_N}(1-e^{-\theta s})-1}{s} & 0 & -\frac{1}{s} & 0 \\ 0 & \frac{k_p}{(s+k_p)} & 0 & 0 & \frac{-k_p}{(s+k_p)} \end{bmatrix} \begin{bmatrix} L \\ V \\ D \\ B \\ V_T \end{bmatrix}$$

- Através do controle de ambas as composições consegue-se para colunas de alta puraze uma redução de até 15% no consumo energético.
- Justamente colunas de alta pureza são as mais não lineare e dependentes da direção do sinal de entrada.
- Portanto, para estas colunas a correta escolha da estrutura de controle desempenha um papel primordial.
- O diagrama de McCabe-Thiele- mostra que R=L/D tem uma direta influência na composição de topo (y_D). Então R ou uma de suas variantes como R/(R+1) é uma forte candidata a ser usada como variável manipulada.



- Segundo as suas propriedades as estruturas de controle são agrupadas em :

_(L,V)

_(D,V) e (L,B)

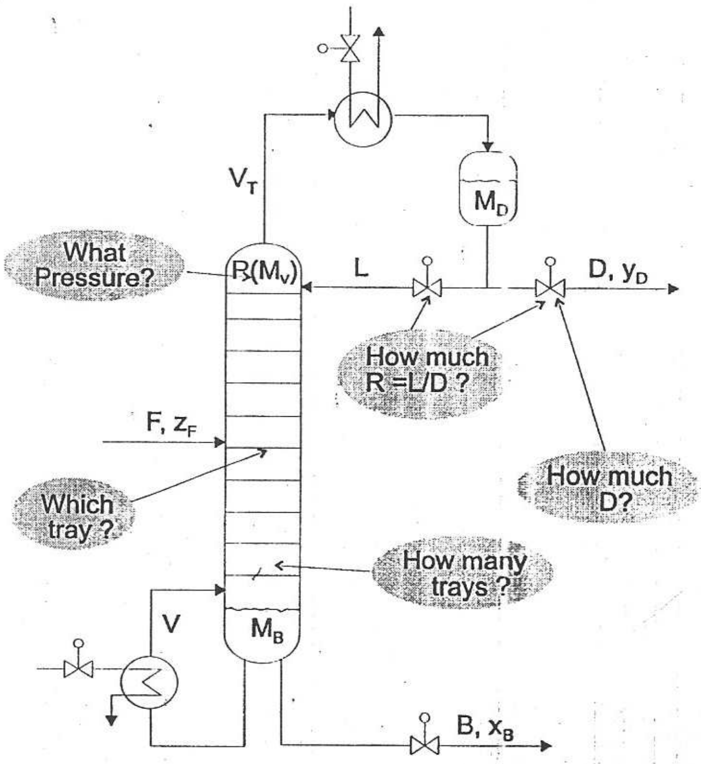
_(D,B)

_(L/D,V), (L/(D+L),V) e (D/(D+L),V)

_(L/D, V/B) e (L/(D+L),V/B)

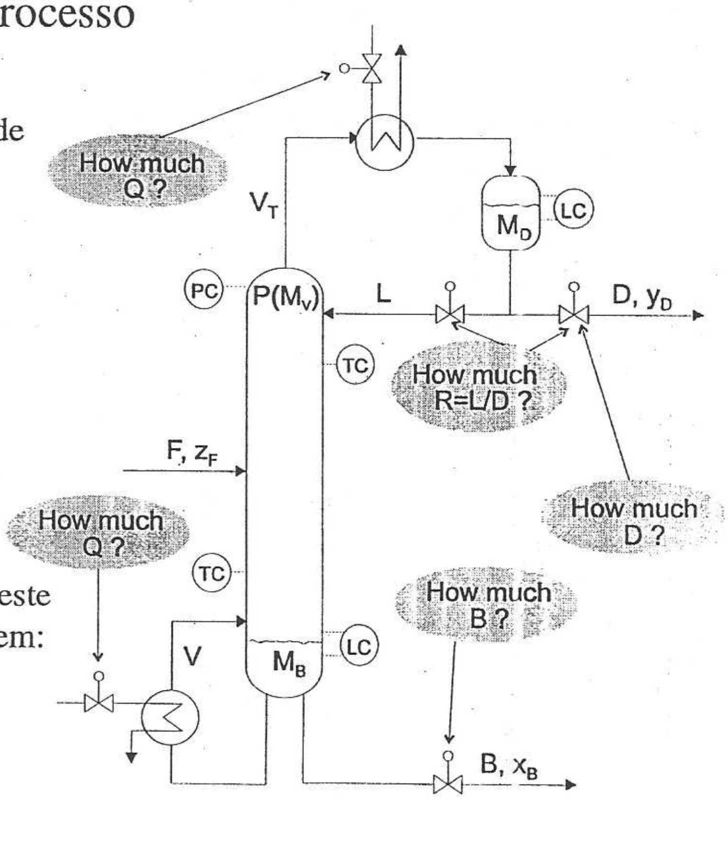
Projeto da Coluna

- O que pode ser especificado durante o dimensionamento de uma coluna de destilação?
- Existem 3 graus de liberdade de operação.
- O objetivo é alcançar uma determinada composição de topo e de fundo. Para isto, R e D (ou B) são normalmente alterados iterativamente até que as composições desejadas sejam alcançadas..



Graus de Liberdade do Processo

- O processo tem 5 graus de liberdade => sistema multivariável (5x5)
- As variáveis controladas são:
  _acúmulos : M_D,M_B, M_V(P)
  _composições: y_D, x_B
- As variáveis manipuladas são: L,D,B,V,V_T
- Os distúrbios são: F, z_F, H_F
- Dada suas propriedades dinâmicas este sistema 5x5 pode ser decomposto em:
  - 3 SISO para o controle dos acúmulos e
  - 1 (2x2) para o controle da composição



$$\underbrace{\begin{bmatrix}\Delta y_D \\ \Delta x_B\end{bmatrix}}_{y} = \underbrace{\begin{bmatrix}0.878 & -0.864 \\ 1.082 & -1.096\end{bmatrix}}_{LV(0)} \underbrace{\begin{bmatrix}\Delta L \\ \Delta V\end{bmatrix}}_{d}$$

$$d_1 = \begin{bmatrix}1\\0\end{bmatrix}, d_2 = \begin{bmatrix}0\\1\end{bmatrix}, d_3 = \begin{bmatrix}0.707\\0.707\end{bmatrix}, d_4 = \begin{bmatrix}0.707\\-0.707\end{bmatrix}, d_5 = \begin{bmatrix}0.6\\-0.8\end{bmatrix}$$

$$y_1 = \begin{bmatrix}0.88\\1.08\end{bmatrix}, y_2 = \begin{bmatrix}-0.86\\-1.10\end{bmatrix}, y_3 = \begin{bmatrix}0.01\\-0.01\end{bmatrix}, y_4 = \begin{bmatrix}1.23\\1.54\end{bmatrix}, y_5 = \begin{bmatrix}1.22\\1.53\end{bmatrix}$$

Decomposição de uma matriz complexa $M_{no \times ni}$ em valores singulares (SVD):

$$M = U\Sigma V^H = \sum_{i=1}^{n} \sigma_i(M) u_i v_i^H$$

- As matrizes U e V são matrizes unitárias com vetores dispostos nas suas colunas

$$U = [\,\overline{u} = u_1,\, u_2, \ldots,\, u_{no} = \underline{u}\,]$$
$$V = [\,\overline{v} = v_1,\, v_2, \ldots,\, v_{ni} = \underline{v}\,]$$

- Σ (valores singulares)

$\Sigma_n = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$;
$n = min\{no, ni\}$

$$\Sigma = \begin{pmatrix}\Sigma_n \\ 0\end{pmatrix} \quad \text{p}/\ no \geq ni$$
$$\Sigma = (\Sigma_n \quad 0) \quad \text{p}/\ no < ni$$

em ordem decrescente, isto é

$$\overline{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n = \underline{\sigma}$$

- valor singular máximo ($\overline{\sigma}$) e mínimo ($\underline{\sigma}$) são os mais importantes.

$$\overline{\sigma}(M) = \max_{u \neq 0} \frac{\|Mu\|_2}{\|u\|_2} \quad e$$
$$\underline{\sigma}(M) = \min_{u \neq 0} \frac{\|Mu\|_2}{\|u\|_2}$$

$$LV(0) = \underbrace{\begin{bmatrix}0.625 & -0.781 \\ 0.781 & 0.625\end{bmatrix}}_{U} \underbrace{\begin{bmatrix}1.972 & 0 \\ 0 & 0.01391\end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix}0.707 & -0.708 \\ -0.708 & -0.707\end{bmatrix}^H}_{V}$$

A.1 SVD (Singular Value Decomposition)

A convenient way of representing a matrix that exposes its internal structure is known as the Singular Value Decomposition (SVD). For a *no* x *ni* complex matrix M, the SVD of M is given by

$$M = U\Sigma V^H = \sum_{i=1}^{n} \sigma_i(M) u_i v_i^H \tag{A.1}$$

where U and V are unitary matrices with column vectors denoted by

$$U = [\,\overline{u} = u_1,\, u_2, \ldots,\, u_{no} = \underline{u}\,] \quad \text{and} \quad V = [\,\overline{v} = v_1,\, v_2, \ldots,\, v_{ni} = \underline{v}\,] \tag{A.2}$$

and Σ contains a diagonal nonnegative definite matrix Σ_n of singular values arranged in descending order:

$$\Sigma = \begin{pmatrix}\Sigma_n \\ 0\end{pmatrix} \text{ if } \quad no \geq ni \quad \text{or} \quad \Sigma = (\Sigma_n \quad 0) \text{ if } no < ni \tag{A.3}$$

and $\quad \Sigma_n = diag(\sigma_1, \sigma_2, \ldots, \sigma_n); \quad n = \min\{no, ni\}$ (A.4)

with $\quad \overline{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n = \underline{\sigma}.$ (A.5)

The maximum ($\overline{\sigma}$) and minimum $\underline{\sigma}$ singular values can alternatively be defined by

$$\overline{\sigma}(M) = \max_{u \neq 0} \frac{\|Mu\|_2}{\|u\|_2} \quad \text{and} \quad \underline{\sigma}(M) = \min_{u \neq 0} \frac{\|Mu\|_2}{\|u\|_2}. \tag{A.6}$$

$\overline{\sigma}$ and $\underline{\sigma}$ can be interpreted geometrically as the least upper bound and the greatest lower bound on the magnification of a vector by the matrix operator M.

***Definition* A.1.1 (Euclidean) Condition Number (γ).** The condition number of a matrix is defined as the ratio

$$\gamma(M) \overset{\Delta}{=} \frac{\overline{\sigma}(M)}{\underline{\sigma}(M)}. \tag{A.7}$$

■

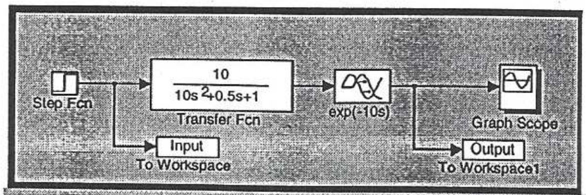
In numerical analysis, the condition number measures the difficulty of inverting a matrix (e.g., [Li93]). It has been argued that it has a control-theoretic significance, in that it measures the inherent difficulty of controlling a given plant. The direct examination of γ alone however gives no conclusive information about the system's controllability, since all systems can be scaled to get a very large (infinite) condition number. On the other hand, the minimal attainable condition number is a finite value that depends on system characteristics only. Thus, for control purposes the minimized condition number over all scaling matrices is more useful to analyze the inherent controllability of the system.

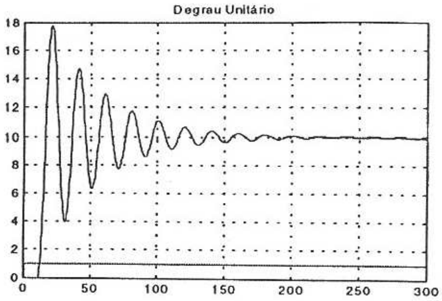
***Definition* A.1.2 Minimized (Euclidean) Condition Number (γ^*).** The minimized condition number is obtained by minimizing the condition number over all possible scalings and is defined by

$$\gamma^*(M) \overset{\Delta}{=} \min_{L,R} \gamma(LMR), \tag{A.8}$$

where L and R are real, diagonal, and nonsingular scaling matrices. If only one side is scaled then we get the input and output minimized condition numbers:

$$\gamma_I^*(M) \overset{\Delta}{=} \min_{R} \gamma(MR) \quad \text{and} \quad \gamma_O^*(M) \overset{\Delta}{=} \min_{L} \gamma(LM). \tag{A.9}$$



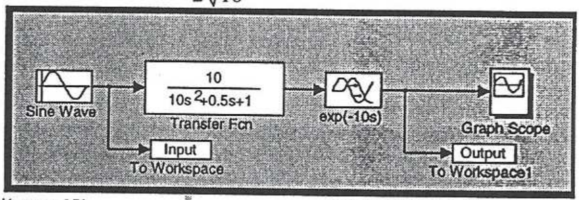


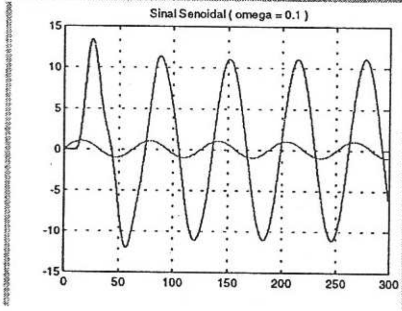
$$\underbrace{P = \frac{2\pi\,\tau}{\sqrt{1-\zeta^2}}}_{Periodo}, \quad \underbrace{OS = exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{Overshoot}, \quad \underbrace{DR = OS^2}_{Razao\ de\ Decaimento}$$

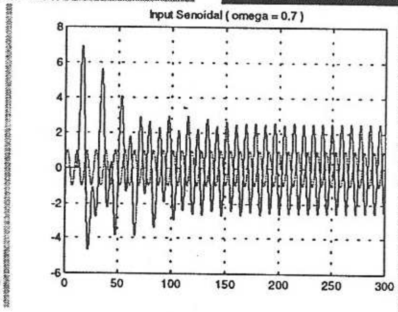
$$y_{ss} = \frac{\overbrace{\frac{AR}{K}}}{\sqrt{\left[1-(\omega\tau)^2\right]^2 + (2\zeta\omega\tau)^2}} A\, sen(\omega t + \phi),$$

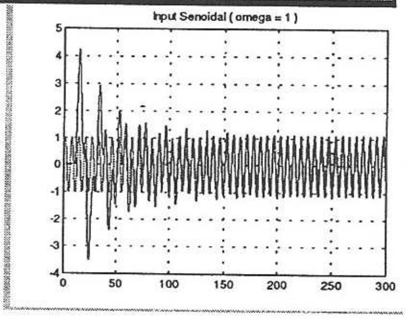
$$\text{onde } \phi = -arctan\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$$

$$\tau = \sqrt{10}\text{ s}, \ \zeta = \frac{0.5}{2\sqrt{10}} = 0{,}079 \quad P = 19{,}93\text{s}, \ OS = 0{,}78$$







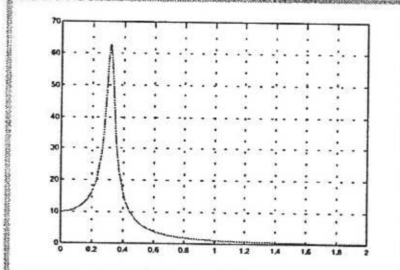


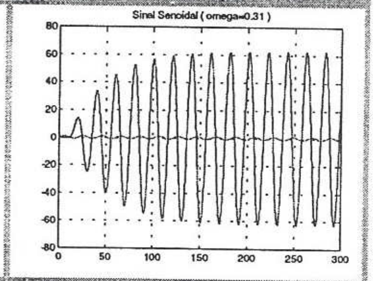
$$\frac{dAR}{d\omega} = 0 \Rightarrow \omega_{max} = \frac{\sqrt{1-2\zeta^2}}{\tau}, \quad \text{para } 0 < \zeta < \frac{1}{\sqrt{2}}$$

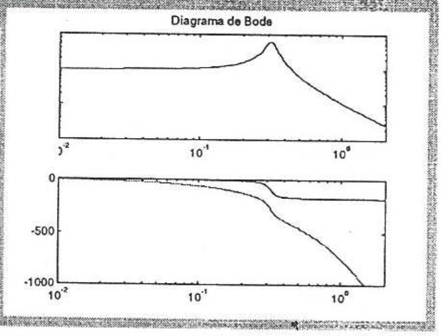
$\omega_{max} = 0{,}314$

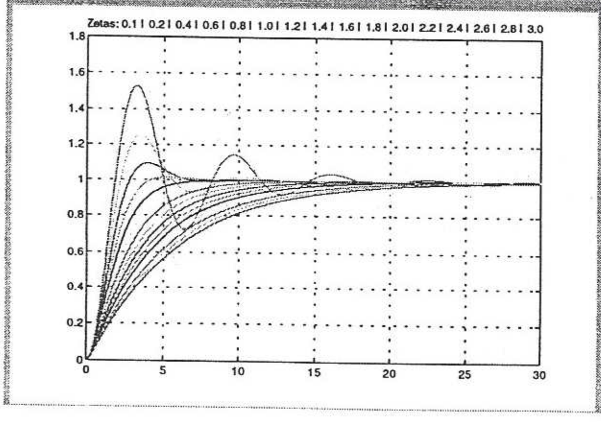
$\omega_{max} = 0{,}314$

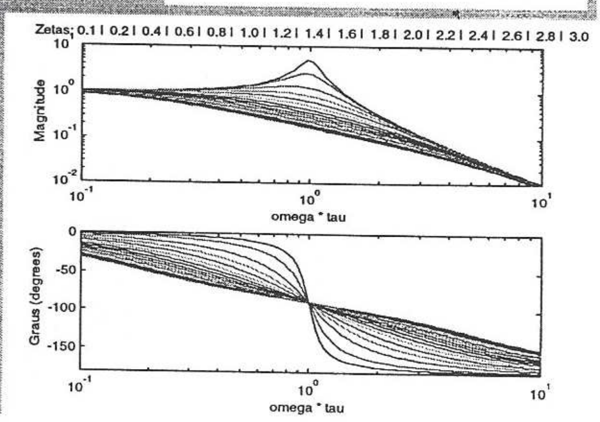
$$y(t) = 63{,}19\, exp(-0.025[t-10])\, cos(0.3153[t-10] + 0{,}5598°) + 63{,}44\, cos(0{,}314[t-10] - 174{,}9°)$$

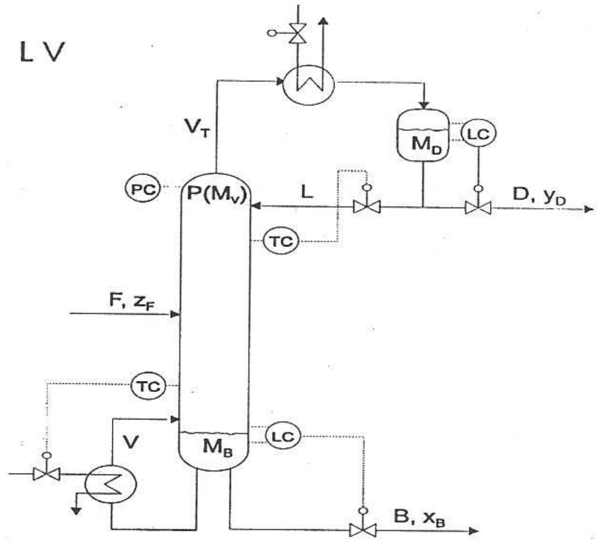


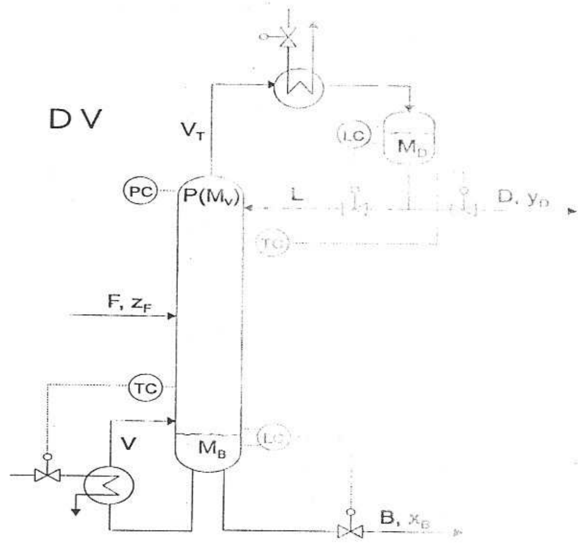






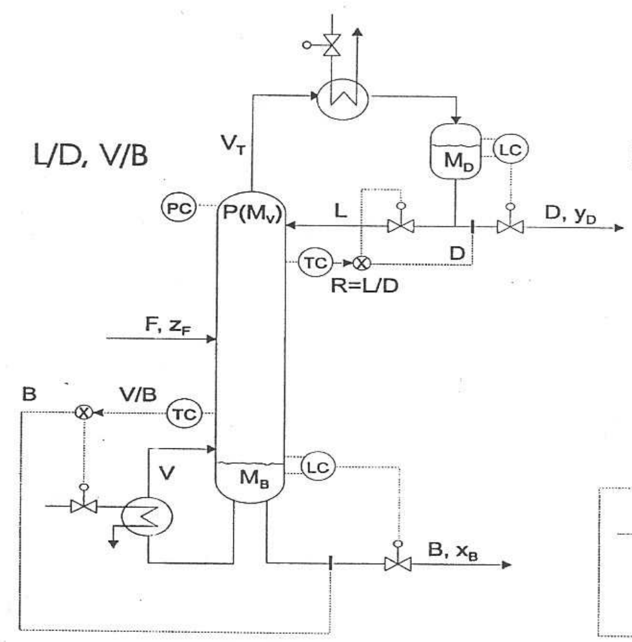


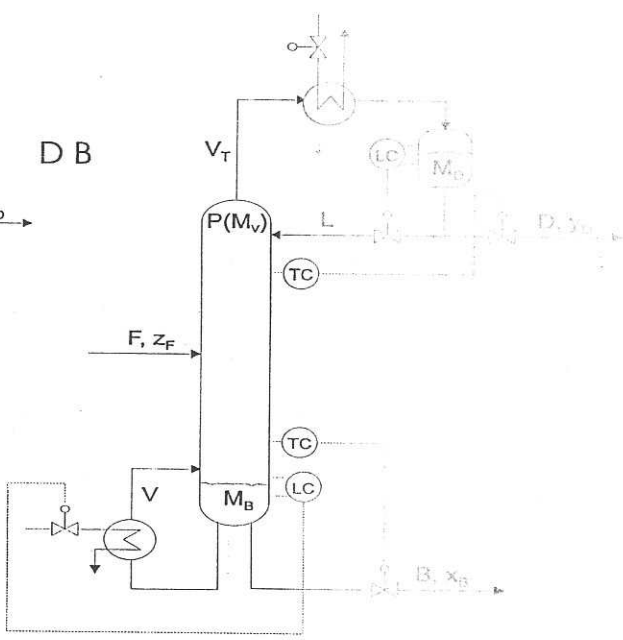




- \+ muito boa para One-point-Control
- \- para L/D > 5, D não pode ser usada para controlar M_D
- \- para colunas de alta pureza (AP) não é recomendada, devido a sua alta sensibilidade a erros na entrada.

- \+ para (AP) não é sensível a erros na entradas
- \+ L/D>5
- \- Ruim, quando $y_D > x_B$ => (L,B)
- • Feedforward para distúrbios em F é recomendável





- \+ muito bom desempenho
- \+ boa capacidade de compensação de distúrbios
- \- precisa de mais medidas

- • só pode ser usada para colunas com elevado número de pratos e em 'dual composition control'

1. Exemplo: CSTR com Reação de Van de Vusse

1.1 Descrição do Processo

A reação de Van de Vusse consiste em uma reação em série e uma em paralelo segundo o seguinte esquema reacional:

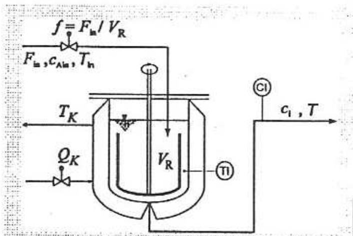
$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$
$$2A \xrightarrow{k_3} D.$$

O objetivo é a produção de B. Os componentes C e D são subprodutos.

Considere que a reação de van de Vusse ocorre em um reator isotérmico idealmente agitado o qual é modelado por:

$$\frac{dc_A}{dt} = \frac{F_{in}}{V_R}\cdot(c_{Ain} - c_A) - [k_1(T)c_A + k_3(T)c_A^2]$$

$$\frac{dc_B}{dt} = -\frac{F_{in}}{V_R}c_B + [k_1(T)c_A - k_2(T)c_B]$$



- c_B : variável controlada
- $f = F_{in}/V_R$: variável manipulada $3 \le f\left[h^{-1}\right] \le 35$
- c_{Ain}, é o principal distúrbio $4.5 \le c_{Ain}[mol/l] \le 5.7.$

$$c_{A0} = \frac{-k_1 - f + \sqrt{(k_1+f)^2 + 4k_3 c_{Ain} f}}{2k_3}$$

$$c_{B0} = k_1 c_{A0}/(k_2 + f)$$

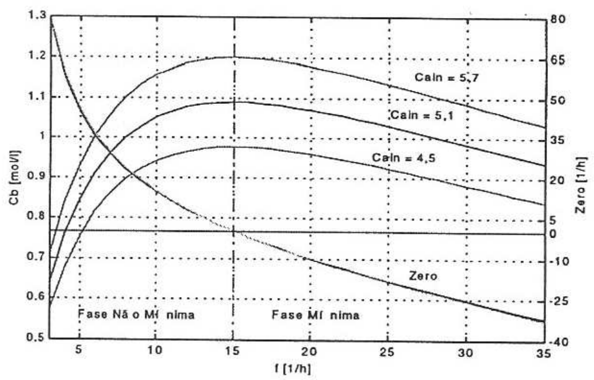
$$\frac{\Delta c_B(s)}{\Delta f(s)} = G(s) = \frac{K(\beta s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

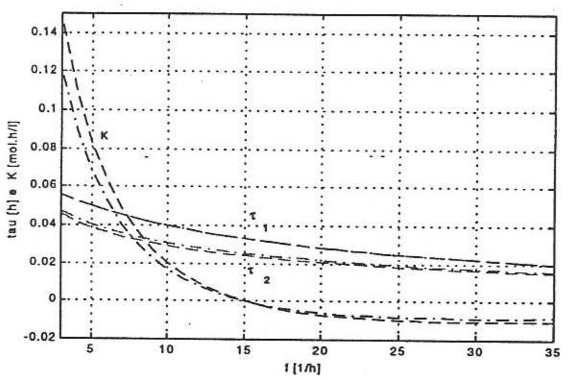
$$\tau_1 = \frac{1}{f_0 + k_1 + 2k_3 c_{A0}},\ \tau_2 = \frac{1}{f_0 + k_2}$$

$$\beta = \frac{-1}{k_1\left(\dfrac{c_{Ain} - c_A}{c_B}\right)_0 - (f_0 + k_1 + 2k_3 c_{A0})}$$

$$f_p = k_2 + (k_1 - k_2)c_{A0}/c_{Ain}$$

$$K = \frac{k_1(c_{Ain} - c_A)_0 - c_{B0}(f_0 + k_1 + 2k_3 c_{A0})}{(f_0 + k_1 + 2k_3 c_{A0})(f_0 + k_2)}$$





Transmission zero

The linearized plant model has a transmission zero that can be computed as

$$z = \frac{-C_{11}(c_{Ain} - c_A)(f + r_2') - C_{12}c_B\left[\dfrac{(c_{Ain} - c_A)}{c_B}r_1' - f - r_1' - r_3'\right]}{C_{11}(c_{Ain} - c_A) - C_{12}c_B}$$

where $r_1' \stackrel{\Delta}{=} k_1(T)$, $r_2' \stackrel{\Delta}{=} k_2(T)$, $r_3' \stackrel{\Delta}{=} 2k_3(T)c_A$, and C_{11} and C_{12} are the elements of the matrix C of the state space model {A,B,C,D} and depend on the measured variables. For example,

- if only c_B is measured, then $C_{11} = 0$ and $C_{12} = 1$;
- if c_A is the output, then $C_{11} = 1$ and $C_{12} = 0$;
- for the case where c_A/c_B are measured, C_{11} and C_{12} are respectively given by $1/c_B$ and $-c_A/c_B^2$.

When only c_B is controlled, the system can exhibit nonminimum phase dynamic behavior (i.e. z > 0).

$$z = \left(\zeta_B^{-1} - 1\right)r_1' - f - r_3'$$

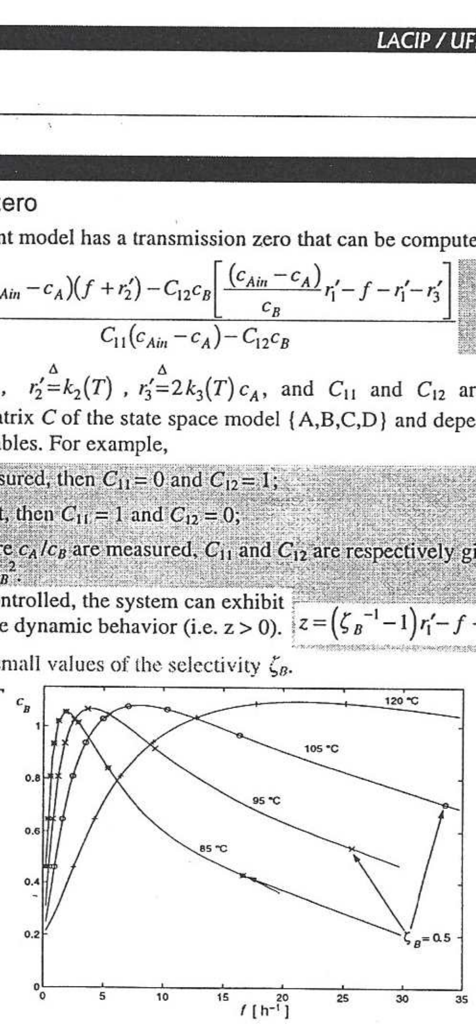
- z is positive for small values of the selectivity ζ_B.

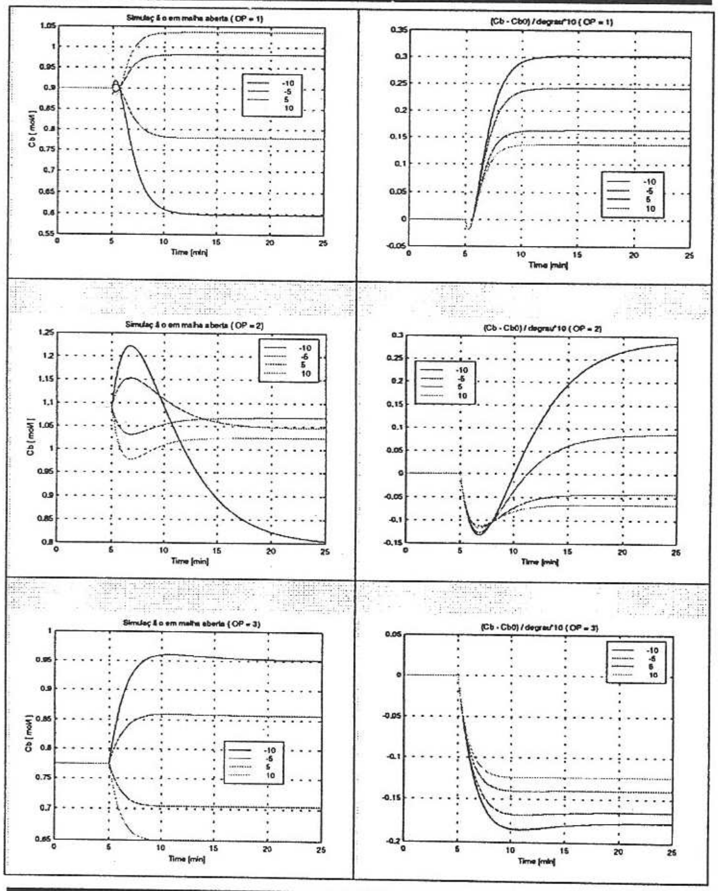
c_B vs. f at several T

The system displays:

- nonminimum phase behavior for OP on the left of the peak and
- minimum-phase behavior for OP on the right

of the peak.





$$\frac{dV_R}{dt}=F_{in}-F_{out}$$

$$\frac{d(V_R c_A)}{dt}=F_{in}\,c_{Ain}-F_{out}\,c_A-V_R\left[k_1(T)c_A+k_3(T)c_A^{\;2}\right]$$

$$\frac{d(V_R c_B)}{dt}=-F_{out}\,c_B+V_R\left[k_1(T)c_A-k_2(T)c_B\right]$$

$$\frac{d(V_R T)}{dt}=F_{in}T_{in}-F_{out}T+\frac{k_W A_R}{\rho C_p}(T_K-T)$$
$$-\frac{V_R}{\rho C_p}\left[k_1(T)c_A\Delta H_1+k_2(T)c_B\Delta H_2+k_3(T)c_A^{\;2}\Delta H_3\right]$$

$$\frac{dT_K}{dt}=\frac{1}{m_K\,C_{pK}}\left[Q_K+k_W A_R(T-T_K)\right].$$

Table A0.1: Chemical kinetic parameters for the Arrhenius equation [EnKI93]

Reaction	Collision factor k_{i0}	Unit of k_{i0}	Activation Energie E_i [K]	Reaction Enthalpy ΔH_i [kJ/mol]
$A\xrightarrow{k_1}B\;\therefore\;r_1=k_1(T)c_A$	$(1.287\pm0.04)\times10^{12}$	h^{-1}	-9758.3	4.2 ± 2.36
$B\xrightarrow{k_2}C\;\therefore\;r_2=k_2(T)c_B$	$(1.287\pm0.04)\times10^{12}$	h^{-1}	-9758.3	$-(11.0\pm1.92)$
$2A\xrightarrow{k_3}D\;\therefore\;r_3=k_3(T)c_A^{\;2}$	$(9.043\pm0.27)\times10^{9}$	$\frac{\text{liter}}{\text{mol h}}$	-8560	$-(41.85\pm1.41)$

For the 3 PDOF problem, where the reactor volume can be manipulated, we need the dependence of the coolant surface A_R on the reactor volume V_R. For a cylindrical reactor, where only the base and the side area are cooled, we can write the relation between A_R and V_R as $A_R=(\pi D_R^2/4)+(4\,V_R/D_R)$.

Table A0.2: Physico-chemical parameters and reactor dimensions [EnKI93]

Parameter Name	Symbol	Value	Unit
density of mixture	ρ	0.9342 ± 0.0004	$\left[\frac{\text{kg}}{\text{l}}\right]$
heat capacity of mixture	C_p	3.01 ± 0.04	$\left[\frac{\text{kJ}}{\text{kg K}}\right]$
heat transfer coefficient for cooling jacket	k_W	4032 ± 120	$\left[\frac{\text{kJ}}{\text{m}^2\text{ h K}}\right]$
surface of cooling jacket	A_R	0.215	[m²]
nominal reator volume	V_R	10	[l]
reactor diameter	D_R	0.2312 or 0.3678$	[m]
'coolant mass	m_K	5.0	[kg]
heat capacity of coolant	C_{pK}	2.0 ± 0.05	$\left[\frac{\text{kJ}}{\text{kg K}}\right]$

$The reactor diameter was not given in [EnKI93]. The values given here are the positive solutions of the equation $A_R=(\pi D_R^2)/4+(4\,V_R)/D_R$ for the nominal values A_R and V_R given in this table.

Table A0.3: Linearized model of the CSTR with the *Van de Vusse* reaction scheme

$$\begin{cases}\dot{x}=A\,x+B_u\,u+B_d\,d+B_r r+B_h h\\ y=Cx+Du\end{cases}$$

where

$$y^T=[\Delta c_B,\Delta T,\Delta V_R]$$
$$x^T=[\Delta c_A,\Delta c_B,\Delta T,\Delta T_K,\Delta V_R]$$
$$u^T=[\Delta F_{in},\Delta Q_K,\Delta F_{out}]$$
$$d^T=[\Delta c_{Ain},\Delta T_{in}]$$
$$r^T=\begin{cases}\left[\frac{\Delta k_{10}}{k_{10}},\frac{\Delta k_{20}}{k_{20}},\frac{\Delta k_{30}}{k_{30}}\right] \text{ or}\\ \left[\frac{\Delta E_1}{T[\text{K}]},\frac{\Delta E_2}{T[\text{K}]},\frac{\Delta E_3}{T[\text{K}]}\right]\end{cases}$$
$$h^T=[\Delta H_1,\Delta H_2,\Delta H_3,\Delta k_W]$$

Definition of the auxiliary variable f and constants :

$$f=\frac{F_{in}}{V_R},\quad C_1=\frac{1}{\rho\,C_p},\quad C_2=\frac{1}{m_K\,C_{pK}},\quad C_3=\frac{\pi D_R^2}{4},\quad C_4=\frac{4}{D_R}$$

Differentials of the reaction rates:

$$r_1'=\frac{dr_1}{dc_A}=k_{10}\exp\left(\frac{E_1}{T[\text{K}]}\right),\quad r_1''=-\frac{dr_1}{dT}=\frac{E_1\,r_1}{(T[\text{K}])^2}$$
$$r_2'=\frac{dr_2}{dc_B}=k_{20}\exp\left(\frac{E_2}{T[\text{K}]}\right),\quad r_2''=-\frac{dr_2}{dT}=\frac{E_2\,r_2}{(T[\text{K}])^2}$$
$$r_3'=\frac{dr_3}{dc_A}=2\,c_A\,k_{30}\exp\left(\frac{E_3}{T[\text{K}]}\right),\quad r_3''=-\frac{dr_3}{dT}=\frac{E_3\,r_3}{(T[\text{K}])^2}$$

$A=$

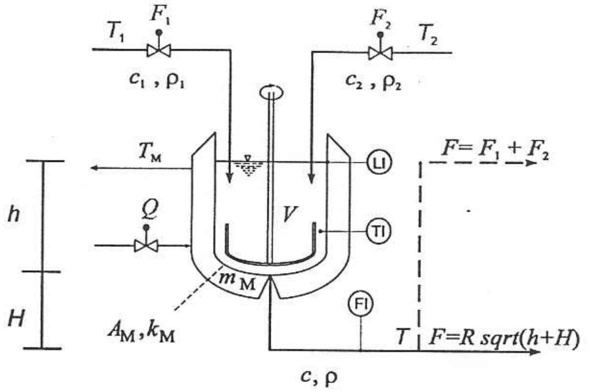
$$\left[\begin{array}{cccc|c}-f-r_1'-r_3' & 0 & r_1''+r_3'' & 0 & -f\,(c_{Ain}-c_A)/V_R\\ r_1' & -f-r_2' & -r_1''+r_2'' & 0 & f\,c_B/V_R\\ -C_1(r_1'\Delta H_1+r_3'\Delta H_3) & -C_1 r_2'\Delta H_2 & \begin{pmatrix}C_1(r_1''\Delta H_1+r_2''\Delta H_2+r_3''\Delta H_3)\\ -f-C_1k_W(C_3+C_4V_R)/V_R\end{pmatrix} & C_1k_W\frac{(C_3+C_4V_R)}{V_R} & \frac{-1}{V_R^{\;2}}\begin{pmatrix}F_{in}(T_{in}-T)+\\ C_1C_3k_W(T_K-T)\end{pmatrix}\\ 0 & 0 & C_2k_W(C_3+C_4V_R) & -C_2k_W(C_3+C_4V_R) & C_2k_W C_4(T-T_K)\\ \hline 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$B_u=\left[\begin{array}{cc|c}(c_{Ain}-c_A)/V_R & 0 & 0\\ -c_B/V_R & 0 & 0\\ (T_{in}-T)/V_R & 0 & 0\\ 0 & C_2 & 0\\ \hline 1 & 0 & -1\end{array}\right],\quad B_r=\left[\begin{array}{ccc}-r_1 & 0 & -r_3\\ r_1 & -r_2 & 0\\ -C_1r_1\Delta H_1 & -C_1r_2\Delta H_2 & -C_1r_3\Delta H_3\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right],\quad C=\left[\begin{array}{cccc|c}C_{11} & C_{12} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1\end{array}\right],$$

$$B_d=\left[\begin{array}{cc}f & 0\\ 0 & 0\\ 0 & f\\ 0 & 0\\ \hline 0 & 0\end{array}\right],\quad B_h=\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -C_1r_1 & -C_1r_2 & -C_1r_3 & C_1\frac{(C_3+C_4V_R)}{V_R}(T_K-T)\\ 0 & 0 & 0 & C_2(C_3+C_4V_R)(T-T_K)\\ \hline 0 & 0 & 0 & 0\end{array}\right],\quad D=\left[\begin{array}{cc|c}0 & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & 0\end{array}\right]$$

Remarks:

1. All variables in the matrices A, B_u, B_d, B_r, B_h, and C correspond to a steady-state solution of the nonlinear model for the physico-chemical parameters of Tables **a0.1** and **a0.2**.
2. The upper left corner in all matrices represents the linearized model corresponding to the 2 PDOF problem. In this case, the system reduces to four equations, i.e., $\Delta V_R=0$.
3. The elements C_{11} and C_{12} of the matrix C depend on the controlled variable. For example, if only c_B is measured and controlled, we will have $C_{11}=0$ and $C_{12}=1$; if c_A is the output, then $C_{11}=1$ and $C_{12}=0$;
for the case where c_A/c_B is measured, C_{11} and C_{12} are given by $1/c_B$ and $-c_A/c_B^2$, respectively; and so on.
4. The set of variables $\left[\frac{\Delta k_{i0}}{k_{i0}}\right]$ and $\left[\frac{\Delta E_i}{T}\right]$ has the same matrix B_r, so they can be analyzed together.



Caso A (V=cte.):

$$\begin{cases} \dfrac{dV}{dt} = F_1 + F_2 - F = 0 \Rightarrow F = F_1 + F_2 \\ V\dfrac{dT}{dt} = F_1(T_1 - T) + F_2(T_2 - T) + \dfrac{Q}{\rho c} \end{cases}$$

$$\begin{cases} \dfrac{dV}{dt} = F_1 + F_2 - F \\ \dfrac{d(VT)}{dt} = F_1T_1 + F_2T_2 - FT + \dfrac{Q}{\rho c} \end{cases}$$

Caso B (V= variável):

$$\begin{cases} F = R\sqrt{h+H},\ se\ H \approx 0 \Rightarrow F = K\sqrt{V} \\ \dfrac{dV}{dt} = F_1 + F_2 - K\sqrt{V} \\ V\dfrac{dT}{dt} = F_1(T_1 - T) + F_2(T_2 - T) + \dfrac{Q}{\rho c} \end{cases}$$

Linearização:

$$\frac{dx}{dt} = f(x,u) \rightarrow \frac{d\Delta x}{dt} = \underbrace{\left[\frac{\partial f}{\partial x}\right]_{x_0,u_0}}_{A} \Delta x + \underbrace{\left[\frac{\partial f}{\partial u}\right]_{x_0,u_0}}_{B} \Delta u$$

Exemplo:

$$x = [V,\ T]^T\ ,\ u_m = [F_1\ F_2\ Q]^T,\ u_d = [T_1\ T_2]^T$$

$$\left[\frac{\partial f}{\partial x}\right]_{x_0,u_0} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial V}\right|_{x_0,u_0} & \left.\dfrac{\partial f_1}{\partial T}\right|_{x_0,u_0} \\ \left.\dfrac{\partial f_2}{\partial V}\right|_{x_0,u_0} & \left.\dfrac{\partial f_2}{\partial T}\right|_{x_0,u_0} \end{bmatrix} = \begin{bmatrix} \dfrac{-K}{2\sqrt{V_0}} & 0 \\ \dfrac{-\left\{ \begin{matrix} F_{10}(T_{10} - T_0) + \\ F_{20}(T_{20} - T_0) + \dfrac{Q_0}{\rho c} \end{matrix} \right\}}{V_0^2} & -\dfrac{F_{10} + F_{20}}{V_0} \end{bmatrix}$$

Solução estacionária:

$$V_0 = \left(\frac{F_{10} + F_{20}}{K}\right)^2 \Rightarrow K = \frac{F_0}{\sqrt{V_0}} \Rightarrow \frac{K}{\sqrt{V_0}} = \frac{F_0}{V_0}\ ,\ \tau \triangleq \frac{V_0}{F_0}$$

$$T_0 = \frac{\left(F_{10}T_{10} + F_{20}T_{20} + \dfrac{Q_0}{\rho c}\right)}{F_{10} + F_{20}} \Rightarrow F_{10}(T_{10} - T_0) + F_{20}(T_{20} - T_0) + \frac{Q_0}{\rho c} = 0$$

Então

$$\left[\frac{\partial f}{\partial x}\right]_{x_0,u_0} = \begin{bmatrix} \dfrac{-F_0}{2V_0} & 0 \\ 0 & -\dfrac{F_0}{V_0} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2\tau} & 0 \\ 0 & \dfrac{-1}{\tau} \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial u_m}\right]_{x_0,u_0} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial F_1}\right|_{x_0,u_0} & \left.\dfrac{\partial f_1}{\partial F_2}\right|_{x_0,u_0} & \left.\dfrac{\partial f_1}{\partial Q}\right|_{x_0,u_0} \\ \left.\dfrac{\partial f_2}{\partial F_1}\right|_{x_0,u_0} & \left.\dfrac{\partial f_2}{\partial F_2}\right|_{x_0,u_0} & \left.\dfrac{\partial f_2}{\partial Q}\right|_{x_0,u_0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \dfrac{(T_{10} - T_0)}{V_0} & \dfrac{(T_{20} - T_0)}{V_0} & \dfrac{1}{\rho c V_0} \end{bmatrix}$$

$$\left[\frac{\partial f}{\partial u_d}\right]_{x_0,u_0} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial T_1}\right|_{x_0,u_0} & \left.\dfrac{\partial f_1}{\partial T_2}\right|_{x_0,u_0} \\ \left.\dfrac{\partial f_2}{\partial T_1}\right|_{x_0,u_0} & \left.\dfrac{\partial f_2}{\partial T_2}\right|_{x_0,u_0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \dfrac{F_{10}}{V_0} & \dfrac{F_{20}}{V_0} \end{bmatrix}$$

Finalmente

$$\frac{d}{dt}\begin{bmatrix} \Delta V \\ \Delta T \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{-1}{2\tau} & 0 \\ 0 & \dfrac{-1}{\tau} \end{bmatrix}}_{A} \overbrace{\begin{bmatrix} \Delta V \\ \Delta T \end{bmatrix}}^{\Delta x} + \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ \dfrac{(T_{10} - T_0)}{V_0} & \dfrac{(T_{20} - T_0)}{V_0} & \dfrac{1}{\rho c V_0} \end{bmatrix}}_{B_m} \overbrace{\begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta Q \end{bmatrix}}^{\Delta u_m} + \underbrace{\begin{bmatrix} 0 & 0 \\ \dfrac{F_{10}}{V_0} & \dfrac{F_{20}}{V_0} \end{bmatrix}}_{B_d} \overbrace{\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix}}^{\Delta u_d}$$

ou equivalentemente

$$\begin{cases} \dfrac{d\Delta V}{dt} = \dfrac{-1}{2\tau}\Delta V + \Delta F_1 + \Delta F_2 \\ \dfrac{d\Delta T}{dt} = \dfrac{-1}{\tau}\Delta T + \dfrac{(T_{10} - T_0)}{V_0}\Delta F_1 + \dfrac{(T_{20} - T_0)}{V_0}\Delta F_2 + \dfrac{1}{\rho c V_0}\Delta Q + \dfrac{F_{10}}{V_0}\Delta T_1 + \dfrac{F_{20}}{V_0}\Delta T_2 \end{cases}$$

Complementando:

$$\begin{bmatrix} \Delta F \\ \Delta T \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{1}{2\tau} & 0 \\ 0 & 1 \end{bmatrix}}_{C} \overbrace{\begin{bmatrix} \Delta V \\ \Delta T \end{bmatrix}}^{\Delta x} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_m} \overbrace{\begin{bmatrix} \Delta F_1 \\ \Delta F_2 \\ \Delta Q \end{bmatrix}}^{\Delta u_m} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{D_d} \overbrace{\begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix}}^{\Delta u_d}$$

Espaço de estado:

$$\begin{cases} \dfrac{d\Delta x}{dt} = A\Delta x + B_m\Delta u_m + B_d\Delta u_d\ ,\quad \Delta x(0) = 0 \\ \Delta y = C\Delta x + D_m\Delta u_m + D_d\Delta u_d \end{cases}$$

Função de Transferência:

$$L\left\{\frac{d\Delta T}{dt} + \frac{1}{\tau}\Delta T = \frac{(T_{10} - T_0)}{V_0}\Delta F_1 + \frac{(T_{20} - T_0)}{V_0}\Delta F_2 + \frac{1}{\rho c V_0}\Delta Q + \frac{F_{10}}{V_0}\Delta T_1 + \frac{F_{20}}{V_0}\Delta T_2\right\}$$

$$s\Delta T(s) + \frac{1}{\tau}\Delta T(s) =$$

$$\frac{(T_{10} - T_0)}{V_0}\Delta F_1(s) + \frac{(T_{20} - T_0)}{V_0}\Delta F_2(s) + \frac{1}{\rho c V_0}\Delta Q(s) + \frac{F_{10}}{V_0}\Delta T_1(s) + \frac{F_{20}}{V_0}\Delta T_2(s)$$

$$\frac{\Delta T(s)}{\Delta F_1(s)} = G_1(s) = \frac{\dfrac{(T_{10} - T_0)}{V_0}}{s + \dfrac{1}{\tau}} = \frac{\tau\dfrac{(T_{10} - T_0)}{V_0}}{\tau s + 1} = \frac{\dfrac{(T_{10} - T_0)}{F_0}}{\tau s + 1} = \frac{K_1}{\tau s + 1}$$

$$\frac{\Delta T(s)}{\Delta F_2(s)} = G_2(s) = \frac{\dfrac{(T_{20} - T_0)}{V_0}}{s + \dfrac{1}{\tau}} = \frac{\tau\dfrac{(T_{20} - T_0)}{V_0}}{\tau s + 1} = \frac{\dfrac{(T_{20} - T_0)}{F_0}}{\tau s + 1} = \frac{K_2}{\tau s + 1}$$

$$\frac{\Delta T(s)}{\Delta Q(s)} = G_3(s) = \frac{\dfrac{1}{\rho c V_0}}{s + \dfrac{1}{\tau}} = \frac{\tau\dfrac{1}{\rho c V_0}}{\tau s + 1} = \frac{\dfrac{1}{\rho c F_0}}{\tau s + 1} = \frac{K_3}{\tau s + 1}$$

Systemas de Segunda Ordem: $G(s) = \dfrac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$

$$\zeta > 1,\quad y(t) = K\left\{1 - exp\left(-\frac{\zeta t}{\tau}\right)\left[cosh\left(\frac{\sqrt{\zeta^2 - 1}}{\tau}t\right) + \frac{\zeta}{\sqrt{\zeta^2 - 1}}senh\left(\frac{\sqrt{\zeta^2 - 1}}{\tau}t\right)\right]\right\}$$

$$\zeta = 1,\quad y(t) = K\left\{1 - \left(1 + \frac{t}{\tau}\right)exp\left(-\frac{t}{\tau}\right)\right\}$$

$$0 \le \zeta < 1,\quad y(t) = K\left\{1 - exp\left(-\frac{\zeta t}{\tau}\right)\left[cos\left(\frac{\sqrt{1 - \zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1 - \zeta^2}}sen\left(\frac{\sqrt{1 - \zeta^2}}{\tau}t\right)\right]\right\}$$