Some Basics of Linear Dynamics

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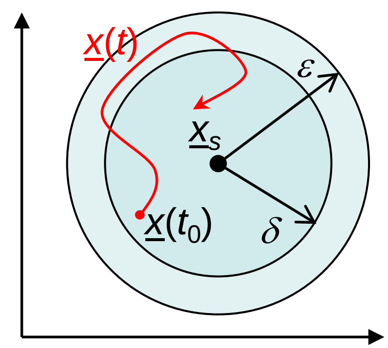
Stability of an Equilibrium Point (Lyapunov, 1898):

Equilibrium Point defined by $\dot{\underline{x}} = \underline{f}(\underline{x}_s, \underline{u}_s) \overset{!}{=} 0$

$\underline{x}_s$ is called stable, if $\forall\ \varepsilon > 0$ there exists a $\delta(\varepsilon)$ such that:

$$\left\|\underline{x}(t_0) - \underline{x}_s\right\| \le \delta(\varepsilon) \quad \Rightarrow \quad \left\|\underline{x}(t) - \underline{x}_s\right\| \le \varepsilon \quad \forall\, t \ge t_0$$

[$\underline{x}(t)$: trajectory, evolution of the state over time]

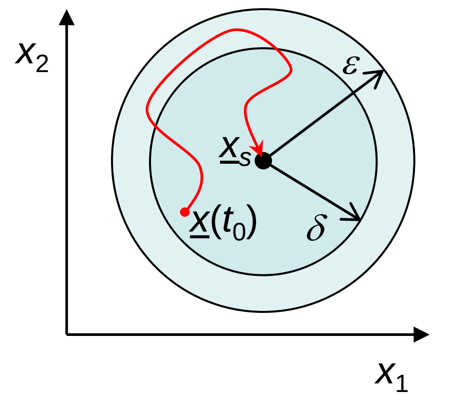


If the initial state $\underline{x}(t_0)$ is sufficiently close to a stable equilibrium $\underline{x}_s$, then the state of the system stays close to $\underline{x}_s$ for all times.

Asymptotic Stability

If in addition: $\lim_{t \to \infty} \underline{x}(t) = \underline{x}_S$

then $\underline{x}_S$ is called asymptotically stable.



If more than one equilibrium exists, each stable equilibrium $\underline{x}_S$ has a domain of attraction, i.e., the trajectories $\underline{x}(t)$ that start inside the domain of attraction stay inside.

If a system has only one stable equilibrium $\underline{x}_S$ and all trajectories $\underline{x}(t)$ end in $\underline{x}_S$, then the system is called globally asymptotically stable.

Linearization around Equilibrium Points

Given: Nonlinear dynamic process model :

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}), \quad \underline{f}(\underline{x}_S, \underline{u}_S) = \underline{0} \Leftrightarrow (\underline{x}_S, \underline{u}_S) \text{ is an equilibrium point}$$

$$\underline{x} \in R^n, \quad \underline{u} \in R^p$$

Wanted: (approximative) behavior around $(\underline{x}_S, \underline{u}_S)$

Approach: $\underline{x}(t) = \underline{x}_S + \Delta\underline{x}(t), \quad \underline{u}(t) = \underline{u}_S + \Delta\underline{u}(t)$

for small deviations $(\Delta\underline{x}(t), \Delta\underline{u}(t))$ from $(\underline{x}_S, \underline{u}_S)$

Taylor-Series Expansion:

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}_S, \underline{u}_S) + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}}_{\underline{J}_{fx}(\underline{x}_S, \underline{u}_S)} \cdot \Delta \underline{x}(t) + \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}}_{\underline{J}_{fu}(\underline{x}_S, \underline{u}_S)} \cdot \Delta \underline{u}(t) + \ldots$$

- Higher order terms are neglected
- The derivatives are evaluated at $(\underline{x}_S, \underline{u}_S)$
- $\underline{f}(\underline{x}_S, \underline{u}_S) = \underline{0}$

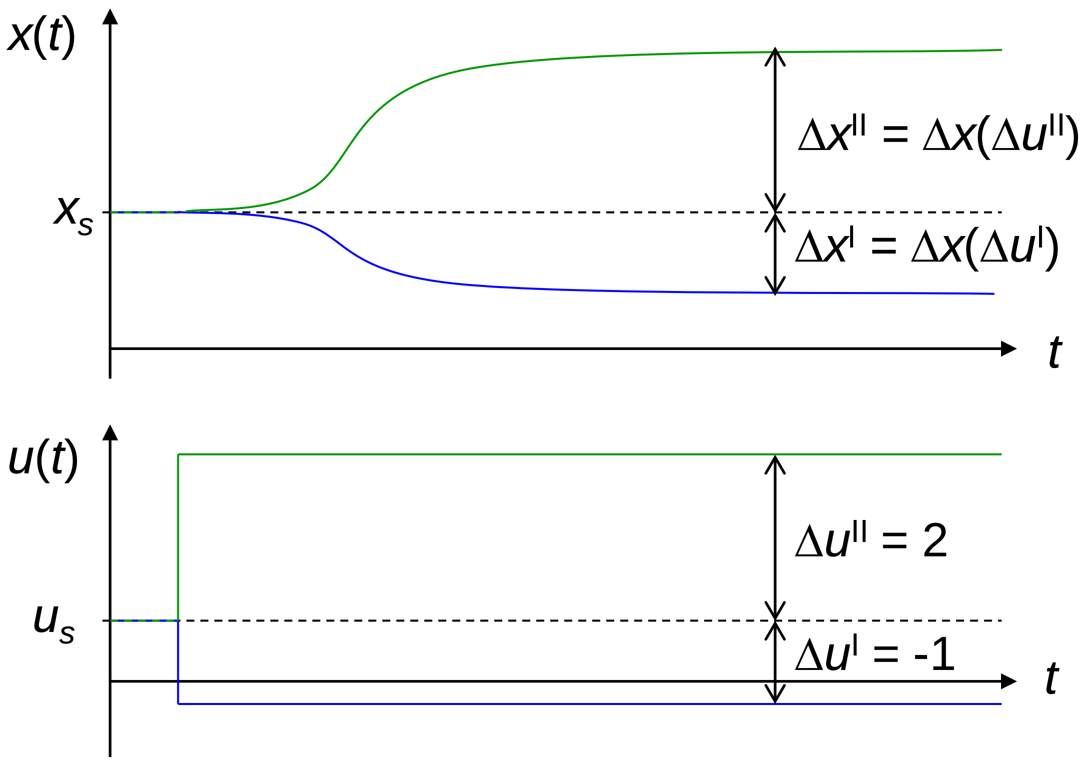
$$\Rightarrow \qquad \underline{\dot{x}}(t) \approx \Delta \underline{\dot{x}}(t) = \underline{J}_{fx}(\underline{x}_S, \underline{u}_S) \cdot \Delta \underline{x}(t) + \underline{J}_{fu}(\underline{x}_S, \underline{u}_S) \cdot \Delta \underline{u}(t)$$

$$=: \underline{A} \cdot \Delta \underline{x}(t) + \underline{B} \cdot \Delta \underline{u}(t)$$

Linearized System

Result: Linear system, where $\underline{A}$, $\underline{B}$ depend on $(\underline{x}_s, \underline{u}_s)$

The validity of the approximation depends on the higher-order terms of the Taylor-series and can often be checked by simulation:



Check $\Delta \underline{x}$ for different $\Delta \underline{u}$ for the nonlinear system, e.g.:

$$\Delta x\left(\Delta u^{I}\right) \overset{?}{\approx} -2 \cdot \Delta x\left(\Delta u^{II}\right)$$

$\rightarrow$ approximation is valid

Dynamic Behavior in around Stationary Points

Autonomous system: $\dot{\underline{x}} = \underline{A} \cdot \underline{x}, \quad \underline{x}(t_0) = \underline{x}_0$

Eigenvalues of $\underline{A}$: $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

If the eigenvalues are real and distinct, n different eigenvectors exist: $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$

Then, $\underline{x}_0$ can be decomposed in the eigendirections:

$$\underline{x}_0 = \gamma_1 \cdot \underline{v}_1 + \gamma_2 \cdot \underline{v}_2 + \ldots + \gamma_n \cdot \underline{v}_n = \underline{V} \cdot \underline{\gamma}$$

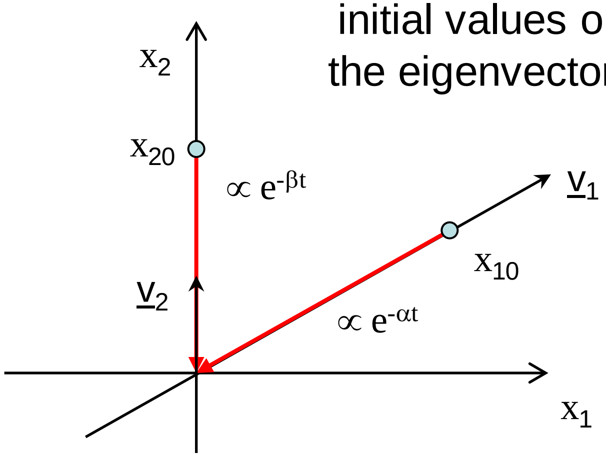
where: $\underline{V} = (\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n)$ and $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$

The trajectory of the state results as

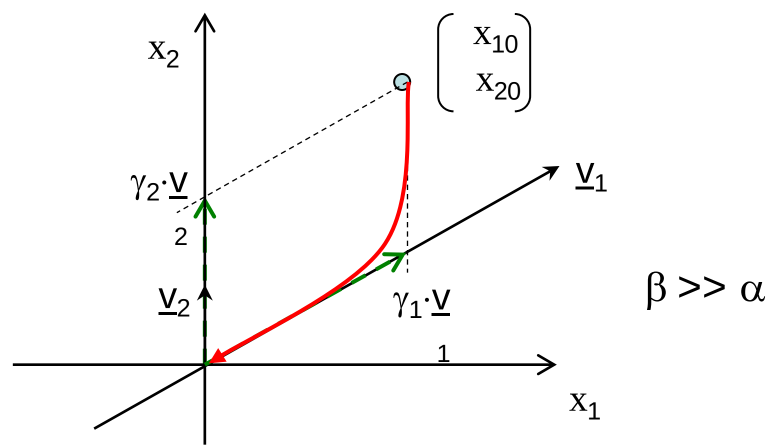
$$\underline{x}(t) = \gamma_1 \cdot e^{\lambda_1 \cdot t} \cdot \underline{v}_1 + \gamma_2 \cdot e^{\lambda_2 \cdot t} \cdot \underline{v}_2 + \ldots + \gamma_n \cdot e^{\lambda_n \cdot t} \cdot \underline{v}_n$$

$$\Leftrightarrow \quad \underline{x}(t) = \underline{V} \cdot \begin{pmatrix} e^{\lambda_1 \cdot t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 \cdot t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n \cdot t} \end{pmatrix} \cdot \underline{\gamma} = \underbrace{\underline{V}}_{\text{super-position}} \cdot \underbrace{diag\left(e^{\lambda_i \cdot t}\right)}_{\text{dynamics}} \cdot \underbrace{\underline{V}^{-1} \cdot \underline{x}_0}_{\text{decomposition}}$$

initial values on the eigenvectors



arbitrary initial values



- If $\beta >> \alpha$: $\quad e^{-\beta \cdot t}$ goes faster to zero than $e^{-\alpha \cdot t}$

Role of the Eigenvalues

Shorter representation: $\underline{x}(t) = e^{\underline{A} \cdot t} \cdot \underline{x}_0$

with the fundamental matrix:

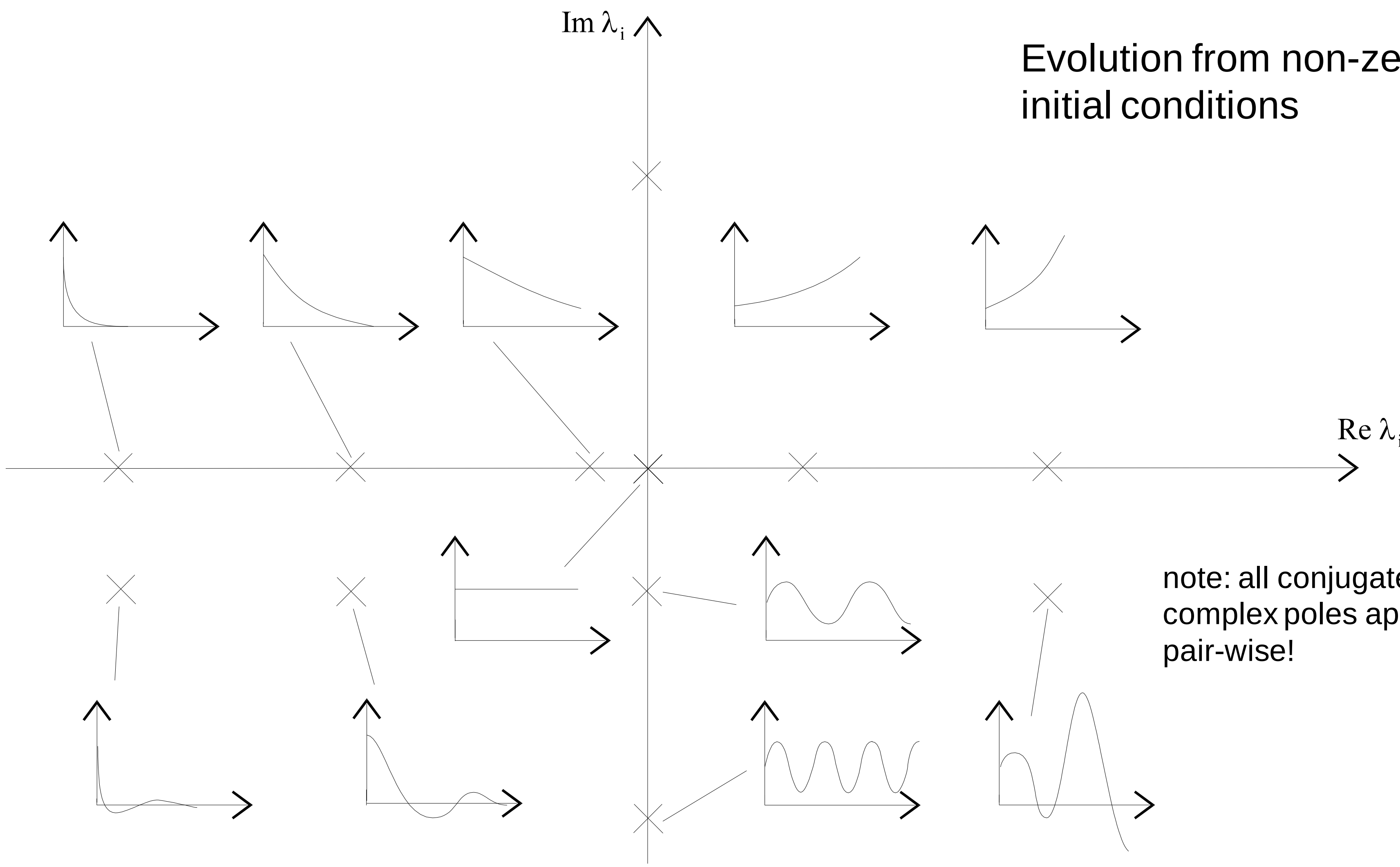
$$e^{\underline{A} \cdot t} = \underline{I} + \underline{A} \cdot t + \frac{1}{2} \cdot \underline{A}^2 \cdot t^2 + \ldots \quad = \underline{V} \cdot \begin{pmatrix} e^{\lambda_1 \cdot t} & \ldots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & e^{\lambda_n \cdot t} \end{pmatrix} \cdot \underline{V}^{-1}$$

Stability and eigenvalues of linear systems:

- If all λ_i have negative real parts: all trajectories converge to $\underline{0}$ $\Rightarrow$ the system is asymptotically stable! $\underline{0}$ is the only equilibrium.
- If one λ_i has a positive real part: all trajectories tend to infinity (in the direction $\underline{v}_i$) $\Rightarrow$ the system is unstable!
- If some eigenvalues have zero real parts and these are simple, and all other ones are negative: all trajecories converge either to points or to limit cycles $\Rightarrow$ the system is stable!

Transient Behavior and Eigenvalues of Linear Systems

Evolution from non-zero initial conditions



note: all conjugate complex poles appear pair-wise!

Stability of Equilibrium Points of Nonlinear Systems

Let $\underline{x}_s$ be an equilibrium point and $\underline{A}(\underline{x}_s)$ the matrix obtained from linearization at $\underline{x}_s$.

- $\underline{x}_s$ is stable, if all eigenvalues of $\underline{A}(\underline{x}_s)$ have negative real parts:

  asymptotic stability of $\underline{x}_s$ $\Leftrightarrow$ asymptotic stability of the behavior of the linearized model around $\underline{x}_s$

- $\underline{x}_s$ is unstable if one real part is positive
- if the real part of an eigenvalue is zero, no statement is possible

Note: The stability range of $\underline{x}_s$ cannot be inferred from the linearized model!

General Stability Criterion According to Lyapunov

Given: $\dot{\underline{x}} = \underline{f}(\underline{x})$, equilibrium point $\underline{x}_s$

If a function $V(\underline{x})$ exists such that:

$$0 = V(\underline{x}_s) \le V(\underline{x}) \le V_{max} \;, \quad \frac{dV}{dt} = \frac{\partial V}{\partial \underline{x}} \dot{\underline{x}} \le 0$$

holds in a region Γ, and $V(\underline{x}) = V_{max}$, then all trajectories with $\underline{x}(0) \in \Gamma$ end in $\underline{x}_s$, i.e., Γ is part of the domain of attraction of x_s.

Difficult: find a suitable Lyapunov function $V(\underline{x})$.

Simplest approach: $V(\underline{x}) = \underline{x}^T \cdot \underline{P} \cdot \underline{x}$ (with a positive definite $\underline{P}$)