1.3 Eigenvalues, Eigenvectors and Controllability

Excerpts from Lecture Notes of the Course Advanced Process Control by Prof. Sebastian Engell, TU Dortmund

LTI - system $\quad \dot{\underline{x}} = \underline{A}\,\underline{x}, \quad \underline{x}(0) = \underline{x}_0$

eigenvalues of $\underline{A}$: $\det(\lambda_i \underline{I} - \underline{A}) = 0$

$$\Leftrightarrow \exists\, \underline{v}_i \text{ such that } \underline{A}\,\underline{v}_i = \lambda_i \underline{v}_i$$

If $\underline{x}_0 = \alpha \cdot \underline{v}_i \Rightarrow \underline{x}(t) = \alpha\, \underline{v}_i\, e^{\lambda_i t} = \underline{x}_0\, e^{\lambda_i t}$

$[\alpha\, \underline{v}_i\, \lambda_i\, e^{\lambda_i t} = \underline{A}\, \alpha\, \underline{v}_i\, e^{\lambda_i t}]$

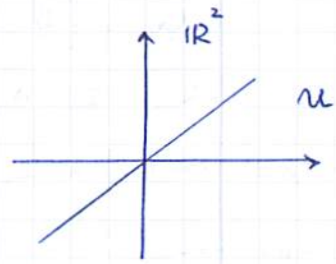
More generally, if all λ_i are distinct then $\underline{x}_0$ can always be written as

$$\underline{x}_0 = \sum_{i=1}^{n} \alpha_i \underline{v}_i \Rightarrow \underline{x}(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \underline{v}_i$$

$$\underline{x}_0 = \underline{V} \cdot \underline{\alpha} \qquad V = [\underline{v}_1 \dots \underline{v}_n] \qquad \underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\underline{x}(t) = \underline{V}\, e^{\Lambda t}\, \underline{\alpha} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

invariant subspaces



subspace $\mathcal{U} \in \mathbb{R}^n$: $\text{span}[\underline{u}_1 \dots \underline{u}_p] \qquad p \le n$

$$= \beta_1 \underline{u}_1 + \beta_2 \underline{u}_2 + \dots + \beta_p \underline{u}_p$$

Dynamic system $\quad \dot{\underline{x}} = \underline{A}\,\underline{x}, \quad \underline{x} \in \mathbb{R}^n$

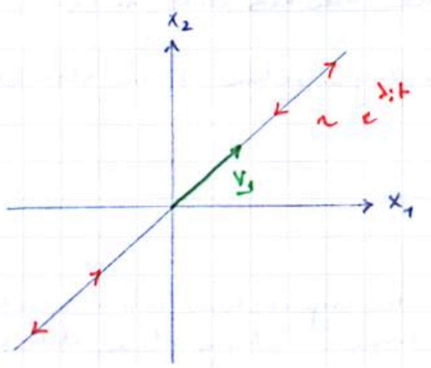
$\mathcal{U}$ is an invariant subspace of the system $\dot{\underline{x}} = \underline{A}\,\underline{x}$

if $\forall\, \underline{x}_0 \in \mathcal{U} \quad \underline{x}(t) \in \mathcal{U}$

[if a trajectory starts in the invariant subspace $\mathcal{U}$, it stays there]

suppose that $\underline{A}$ has n distinct eigenvalues, hence n independent eigenvectors $\underline{v}_i$ then all invariant subspaces of dimension p are spanned by p eigenvectors.

All 1-dimensional subspaces are of the form $\alpha \cdot \underline{v}_i$

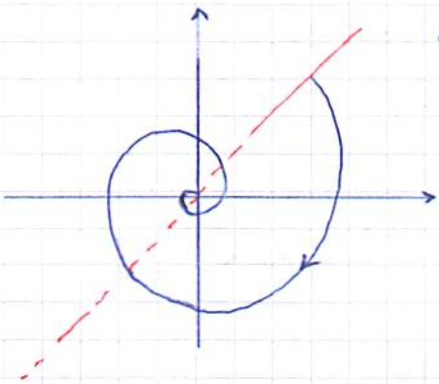


λ_i determine the speed of convergence to the equilibrium.

Conjugate Complex eigenvalues: invariant subspaces are two dimensional

$\lambda_{1,2} = \alpha \pm j\beta$

$\alpha < 0$



real part < 0

Controlability

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \qquad \underline{u} \in \mathbb{R}^p \qquad x \in \mathbb{R}^n$$

$(\underline{A}, \underline{B})$ is called Controllable if $\forall x_0 \; \exists x \in \mathbb{R}^n \; \underline{u}(t, x_0)$ such that $\underline{x}(t) \Rightarrow \underline{0} \; \forall \tau > 0$

Criterion to check Controllability:

$$\underline{Q}_c = \left[\underline{B} \;\; \underline{A}\,\underline{B} \; \dots \; \underline{A}^{n-1}\underline{B} \right] \qquad \text{rank}(\underline{Q}_c) = n \qquad \text{Kalman Criterion}$$

or $\quad \text{rank}\left[s\underline{I} - \underline{A} \,;\, \underline{B} \right] = n \quad \forall s \in \mathbb{C}$

$\iff \text{rank}\left[\lambda_i \underline{I} - \underline{A} \,;\, \underline{B} \right] = n$ for all eigenvalues λ_i of $\underline{A}$ Hautus criterion.

Importance: Controlability is necessary and sufficient for eigenvalue assignment

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$

state feedback : $\underline{u} = \underline{K}\,\underline{x}$ linear proportional feedback.

$$\Rightarrow \dot{\underline{x}} = (\underline{A} + \underline{B}\,\underline{K})\,\underline{x}$$

if $(\underline{A}, \underline{B})$ is Controllable. then for all chosen eigenvalues.

$\lambda_1 \dots \lambda_n$ (real or Conjugate Complex) there is a matrix $\underline{K}$ such that

$\underline{A} + \underline{B}\,\underline{K}$ has these eigenvalues.

suppose λ_k is an eigenvalue of $\underline{A}$ for which the Hautus criterion fails

(uncontrollable eigenvalue). Then λ_k always is an eigenvalue of the closed loop

system $\underline{A} + \underline{B}\,\underline{K}$.

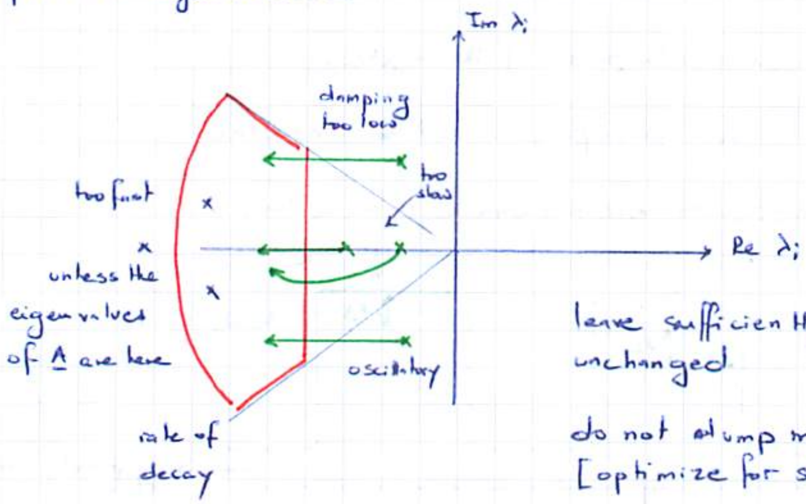
$$\{\lambda_i \text{ of } \underline{A}\} = \{\lambda_i\} + \underbrace{\{\lambda_{nc}\}}_{\text{uncontrollable eigenvalues}}$$

$$\{\lambda_i \text{ of } \underline{A} + \underline{B}\,\underline{K}\} = \{\lambda_{free}\} + \{\lambda_{nc}\}$$

uncontrollable eigenvalues remains where they are. Cannot be changed by state feedback

when is it possible to stabilize a linear system ?

$\rightarrow$ if all unstable eigenvalues are Controllable ["stabilizable"] where to put the eigenvalues?



leave sufficiently fast eigenvalues unchanged

do not lump many eigenvalues in one point
[optimize for small entries in $\underline{k}$]

why not lump? because sensitive to errors.

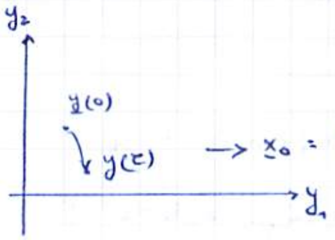
1.4 Observability:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \qquad \underline{u} = \underline{0}$$

$$\underline{y} = \underline{C}\,\underline{x} \qquad \text{measurement equation}, \quad \dim y = m < n$$

Compute $\underline{x}$ from $\underline{y}$! $\quad \underline{x} = \underline{C}^{-1}\underline{y}$ (not possible) $\quad \underline{C} = [\;\overset{n}{}\;]m$

(A, C) is observable if from $y(t)$, $0 \le t \le \tau$, $\underline{x}(0)$ can be uniquely determined for all $\tau > 0$.



$\to \underline{x}_0 = \underline{x}(0)$ once $\underline{x}(0)$ is known, $\underline{x}(t)$ can be computed

Criteria:

$$\underline{Q}_o = \begin{bmatrix} \underline{C} \\ \underline{C}\,\underline{A} \\ \vdots \\ \underline{C}\,\underline{A}^{n-1} \end{bmatrix} \qquad \text{Condition: } \operatorname{rank} \underline{Q}_o = n$$

Kalmann

$$\operatorname{rank} \begin{bmatrix} \lambda_i \underline{I} - \underline{A} \\ \underline{C} \end{bmatrix} = n \quad \forall \text{ eigenvalues of } \underline{A}$$

Hautus

suppose the Hautus criterion does not hold for eigenvalue λ_i

then $\exists\, \underline{v}_i$: $\quad (\lambda_i \underline{I} - \underline{A})\,\underline{v}_i = 0 \to$ eigenvector corresponding to λ_i

and $\quad \underline{C}\,\underline{v}_i = 0 \longrightarrow$ movement along this eigenvector does not affect $\underline{y}$

$$\underline{x}_0 = \alpha\,\underline{v}_i \Rightarrow \underline{x}(t) = \alpha\, e^{\lambda_i t}\,\underline{v}_i$$

$$\underline{y}(t) = 0 \quad \forall t$$

3.2: Kalman filter:

stochastic system:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{v}(t) \qquad \underline{x}(\underline{0}) = \underline{x}_0$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{w}(t)$$

$\underline{w}(t)$: stochastic measurement error

$\underline{v}(t)$ = state noise (disturbances, model errors, ...)

$\underline{v}(t)$, $\underline{w}(t)$: white Gaussian noise : $v_i(t)$ and $v_i(t+\tau)$ are independent for $\tau \neq 0$
no inner structure.

$$\left.\begin{array}{ll} \underline{E}[\underline{v}\,\underline{v}^T] = \underline{Q} & q_{ii} : \text{Cov of } v_i(t) \\ \underline{E}[\underline{w}\,\underline{w}^T] = \underline{R} & r_{ii} : \text{Cov of } w_i(t) \end{array}\right\} \text{Q, R describe the strength of } \underline{V} \text{ resp } \underline{W}$$

task: Build an estimator that minimizes

$$\underline{E}[(\underline{x}-\hat{\underline{x}})^T(\underline{x}-\hat{\underline{x}})] = \underline{E}\sum_{i=1}^{n}(x_i-\hat{x}_i)^2 \qquad E[\;]\text{ expectation } (\approx \text{mean value})$$

least mean squared error estimation.

Solution is "nice" → observer structure.

$$\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{B}\,\underline{u} + \underline{L}(\underline{y} - \underline{C}\,\hat{\underline{x}})$$

$$\underline{L} = \underline{P}\,\underline{C}^T\,\underline{R}^{-1} \qquad \text{(error feedback gain)}$$

$$\dot{\underline{P}}(t) = \underline{A}\,\underline{P} + \underline{P}\,\underline{A}^T + \underline{Q} - \underline{P}\underline{C}^T\underline{R}^{-1}\underline{C}\,\underline{P} \qquad \text{Riccati differential equation}$$

$\underline{P}(t)$ describes the error covariance, $\underline{P}(0)$ is a design parameter Covariance of the error of initial state.

→ same solution applies for time-varying parameters $\underline{A}(t)$, $\underline{B}(t)$, $\underline{C}(t)$, $\underline{Q}(t)$, $\underline{R}(t)$

All parameters Constant, $t \to \infty$

steady state solution $\dot{\underline{P}}(t) = \underline{0}$ → algebraic matrix Riccati equation.

Design problem : choice of $\underline{Q}$ and $\underline{R}$

$\underline{R}$ is more or less given by the data on the sensors.

$\underline{Q}$ is a tuning parameter

"symbolically" $\underline{R}/\underline{Q}$ ↑ then more weight is put on the model

estimator becomes a simulator

$\underline{R}/\underline{Q}$ ↓ then the estimator becomes faster, in the limit $\underline{R} \to \underline{0}$

the measurements are differentiated.

3.3: Observers for systems with unknown inputs:

$$\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{E}\,\underline{d} \qquad \underline{d} \text{ distance, unknown.}$$

E.g:

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & -1 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \underline{u} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} d$$

by $\underline{u} = \underline{K}\,\underline{x}$ it was possible to keep x_3 at zero despite d

Possible with measurement of only some states, in particular only of x_1

($x_3 \equiv 0$ does not give any information)

$\rightarrow$ full theory for this problem

Condition for the existance of observers in state space form are very restrictive

Approximate solution

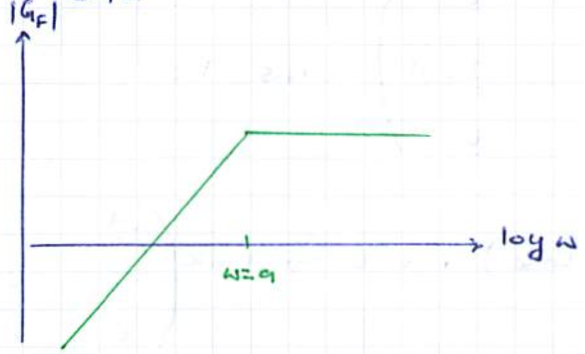
$$\dot{x}_1 = -x_1 + x_2 + u_1$$

x_2 could be estimated exactly if we knew $\dot{x}_1(t)$

$$x_2(t) = x_1(t) + \dot{x}_1(t) - u_1(t)$$

$\dot{x}_1$ can be approximately computed as the output of a linear filter with transfer function

$$G_F(s) = \frac{as}{s+a} \qquad a \text{ ~~small~~ large enough}$$



More realistic approach: $\underline{d}(t)$ is generated by an autonomous model:

$$\underline{\dot{d}} = \underline{H}\,\underline{d} \qquad \underline{d}(0) \text{ unknown} \qquad \dim d = s$$

E.g $\underline{H} = \underline{0}$

$$\underline{\dot{d}} = \underline{0} \rightarrow d_i(t) = \text{constant}$$

$$\underline{\dot{d}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{d} \qquad \dot{d}_1 = d_2 \qquad d_1 = \int d_2$$

$$d_2 = \text{const} = d_{20} \rightarrow d_1(t) = d_{20} \cdot t$$

$d_1(t)$ is a rampe of unknown slope and initial value.

Similarly

$$\underline{H} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \rightarrow \text{generates sinusoids.}$$

Overall model:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{E}\,\underline{d}$$

$$\dot{\underline{d}} = \underline{H}\,\underline{d}$$

$$\begin{pmatrix} \dot{\underline{x}} \\ \dot{\underline{d}} \end{pmatrix} = \begin{pmatrix} \underline{A} & \underline{E} \\ \underline{0} & \underline{H} \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{d} \end{pmatrix} + \begin{pmatrix} \underline{B} \\ \underline{0} \end{pmatrix} \underline{u} \;, \quad \begin{pmatrix} \underline{x} \\ \underline{d} \end{pmatrix}(0) \text{ are unknown}$$

$\dim \underline{y} = m$ $\quad \underline{y} = [\underline{C} \;\; \underline{0}] \begin{pmatrix} \underline{x} \\ \underline{d} \end{pmatrix}$

→ idea: build a Luenberger observer or Kalman filter for the augmented system.

$$\begin{pmatrix} \dot{\hat{\underline{x}}} \\ \dot{\hat{\underline{d}}} \end{pmatrix} = \begin{pmatrix} \underline{A} & \underline{E} \\ \underline{0} & \underline{A} \end{pmatrix} \begin{pmatrix} \hat{\underline{x}} \\ \hat{\underline{d}} \end{pmatrix} + \begin{pmatrix} \underline{B} \\ \underline{0} \end{pmatrix} \underline{u} + \begin{pmatrix} \underline{L}_1 \\ \underline{L}_2 \end{pmatrix} (\underline{y} - \underline{C}\,\hat{\underline{x}})$$

Augmented system must be observable (in particular $\underline{d}$)

Hautus: $\begin{pmatrix} s\underline{I} - \tilde{\underline{A}} \\ \tilde{\underline{C}} \end{pmatrix}$ must have rank $n+s$

$$\text{rank} \begin{pmatrix} s\underline{I} - \underline{A} & \underline{E} \\ \underline{0} & s\underline{I} - \underline{H} \\ \underline{C} & \underline{0} \end{pmatrix} = n+s \quad \forall s$$

$$\underline{H} = \underline{0} \text{ (usual case)}, \; s = 0 \Rightarrow \text{rank} \begin{pmatrix} -\underline{A} & \underline{E} \\ \underline{0} & \underline{0} \\ \underline{C} & \underline{0} \end{pmatrix} = n+s \leq n+m$$

$\dim \underline{d}$ must not be larger than m. if $s \leq m$ and $(\underline{A}, \underline{C})$ is observable then this scheme works.

→ arbitrarily fast convergence is possible in principle.

3.4 Non linear state estimation

system: $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \qquad \underline{x}(0) = \underline{x}_0$

Measurement equation: $\underline{y} = \underline{h}(\underline{x}) \qquad \dim \underline{y} = m < n = \dim \underline{x}$

task: Compute $\underline{x}(t)$ from $\underline{y}(t)$

General estimator: $\dot{\hat{\underline{x}}} = \underline{L}(\hat{\underline{x}}, \underline{u}, \underline{y})$

Want: If $\hat{\underline{x}}(0) = \underline{x}(0) \Rightarrow \hat{\underline{x}}(t) = \underline{x}(t) \quad \forall t > t_0$ — simulator property

$\lim_{t \to \infty} \left(\underline{x}(t) - \hat{\underline{x}}(t) \right) = \underline{0}$ — Convergence.

Simulator: $\underline{L}(\hat{\underline{x}}, \underline{u}, \underline{y}) = \underline{f}(\hat{\underline{x}}, \underline{u})$ if $\underline{h}(\hat{\underline{x}}) = \underline{y}$

Natural: $\underline{L}(\hat{\underline{x}}, \underline{u}, \underline{y}) = \underline{f}(\hat{\underline{x}}, \underline{u}) + \underline{\ell}(\hat{\underline{x}}, \underline{u}, \underline{y} - h(\hat{\underline{x}}))$; $\underline{\ell}(\hat{\underline{x}}, \underline{u}, \underline{0}) = \underline{0}$

Error dynamics: $\underline{e} = \underline{x} - \hat{\underline{x}}$

$$\dot{\underline{e}} = \underline{f}(\underline{x}, \underline{u}) - \underline{f}(\hat{\underline{x}}, \underline{u}) - \underline{\ell}(\hat{\underline{x}}, \underline{u}, \underline{y} - h(\hat{\underline{x}}))$$

$\rightarrow$ in general, error dynamics depend on $\underline{u}, \underline{x}, \hat{\underline{x}}$, not autonomous.

Goal: Design $\underline{\ell}(\hat{\underline{x}}, \underline{u}, \underline{y} - h(\hat{\underline{x}}))$ such that $\underline{e}(t) \underset{t\to\infty}{=} \underline{0}$

For special systems structures $\underline{\ell}$ can be computed such that convergence is guaranteed

No general recipe.

Different options to build an observer:

1. $\underline{\ell}(\hat{\underline{x}}, \underline{u}, \underline{y} - \underline{h}(\hat{\underline{x}})) = \underline{L} \cdot (\underline{y} - \underline{h}(\hat{\underline{x}}))$

$$\dot{\underline{e}} = \underline{f}(\underline{x}, \underline{u}) - \underline{f}(\hat{\underline{x}}, \underline{u}) - \underline{L}(\underline{y} - \underline{h}(\hat{\underline{x}}))$$

$$\dot{\underline{e}} = \underline{f}(\hat{\underline{x}} + \underline{e}, \underline{u}) - \underline{f}(\hat{\underline{x}}, \underline{u}) - \underline{L}(\underline{h}(\hat{\underline{x}} + \underline{e}) - \underline{h}(\hat{\underline{x}})) \qquad \underline{x} = \underline{e} + \hat{\underline{x}}$$

if the system and the observer are linearized around a steady state $(\underline{x}_s, \underline{u}_s)$

$$\dot{\underline{e}} = \left(\underline{A}(\underline{x}_s, \underline{u}_s) - \underline{L}\,\underline{C}(\underline{x}_s) \right) \underline{e}$$

$$\underline{A}(\underline{x}_s, \underline{u}_s) = \left(\frac{\partial f_i}{\partial x_j} \right)_{\underline{x}_s, \underline{u}_s} \qquad \underline{C}(\underline{x}_s) = \left(\frac{\partial h_i}{\partial x_j} \right)_{\underline{x}_s}$$

$\rightarrow$ choose $\underline{L}$ such that $\underline{A} - \underline{L}\,\underline{C}$ is stable.
($\underline{x} - \underline{x}_s$ and $\hat{\underline{x}} - \underline{x}_s$ should be small)

2. adapt the observer to the actual state

$\underline{L} = \underline{L}(\hat{\underline{x}}) \rightarrow \underline{L}$ varies with the estimated state vector

gain scheduled observer

Design by assigning the eigenvalues of $[\underline{A}(\hat{\underline{x}}) - \underline{L}(\hat{\underline{x}})\,\underline{C}(\hat{\underline{x}})]$ at fixed values

$$\underline{A}(\hat{\underline{x}}, \underline{u}) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\underline{x} = \hat{\underline{x}},\, \underline{u}}$$

3- High gain observer (design following Thau (?))

Idea: $\underline{f}(\underline{x}, \underline{u}) = \underbrace{\underline{A}\,\underline{x} + \underline{B}\,\underline{u}}_{\text{linear}} + \underbrace{\underline{\alpha}_1(\underline{x}, \underline{u})}_{\text{non linear}} + \alpha_2(\underline{u}, \underline{y})$

Observer: $\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{B}\,\underline{u} + \underline{\alpha}_1(\hat{\underline{x}}, \underline{u}) + \underline{\alpha}_2(\underline{u}, \underline{y}) + \underline{L}\,(\underline{y} - \underline{h}(\hat{\underline{x}}))$

$$\dot{\underline{e}} = \underline{A}\,\underline{e} + \underline{L}\,(\underline{y} - \hat{\underline{y}}) + \underline{\alpha}_1(\underline{x}, \underline{u}) - \underline{\alpha}_1(\hat{\underline{x}}, \underline{u})$$

Assume : $\underline{h}(\underline{x}) = \underline{C}\,\underline{x}$ (not very restrictive, redefine the states)

$$\dot{\underline{e}} = (\underline{A} - \underline{L}\,\underline{C})\,\underline{e} + \underline{\alpha}_1(\underline{x}, \underline{u}) - \underline{\alpha}_1(\hat{\underline{x}}, \underline{u})$$

If $\overbrace{\|\underline{\alpha}_1(\underline{x}, \underline{u}) - \underline{\alpha}_1(\hat{\underline{x}}, \underline{u})\|}^{\text{degree of non linearity}} \leq \gamma \cdot \|\underline{x} - \hat{\underline{x}}\|$

then $\underline{L}$ can always be chosen such that the eigenvalues of $\underline{A} - \underline{L}\,\underline{C}$ are always large compared to γ and the error dynamics are stable.

Disadvantage: measurement noise is amplified by $\underline{L}$ ("high gain")

4- Extended Kalmann Filter

$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) + \underline{w}$ $\qquad$ $\underline{w}$ is Gaussian white noise covariance $\underline{Q}$

$\underline{y} = \underline{h}(\underline{x}) + \underline{v}$ $\qquad$ $\underline{v}$ " " " " " $\underline{R}$

Estimator: $\dot{\hat{\underline{x}}} = \underline{f}(\underline{x}, \underline{u}) + \underline{L}(t)\,(\underline{y} - \underline{h}(\hat{x}))$

$$\left.\begin{aligned} &\underline{L}(t) = \underline{P}(t)\,\underline{C}^T \cdot \underline{R}^{-1} \\ &\dot{\underline{P}}(t) = \underline{A}\,\underline{P} + \underline{P}\,\underline{A}^T + \underline{Q} - \underline{P}\,\underline{C}^T\,\underline{R}^{-1}\,\underline{C}\,\underline{P} \end{aligned}\right\}\ \text{Kalmann Filter equations}$$

$\underline{A} = \left(\frac{\partial f_i}{\partial x_j}\right)_{\hat{\underline{x}}, \underline{u}}$ $\qquad$ $\underline{C} = \left(\frac{\partial h_i}{\partial x_j}\right)_{\hat{\underline{x}}}$ $\qquad$ linearization at the present estimates (and inputs)

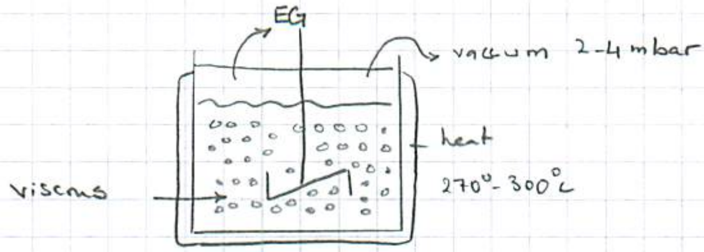
Estimation of system parameters:

$\dot{x} = p\,x + b\,u$ $\qquad$ p varying (slowly)

$\dot{p} = 0$ $\longrightarrow$ non linear system $\longrightarrow$ x and p can be estimated by an EKF

$y = x$

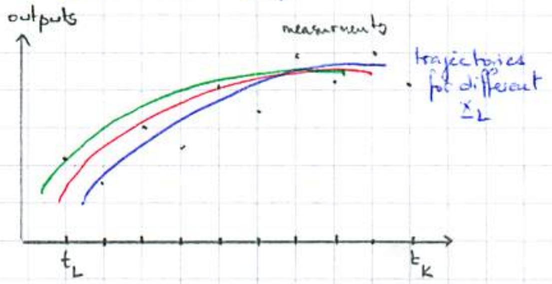
state and parameter estimation for the production of PET



8.5: Moving horizon estimator:

All estimators used so far are recursive:

$$\underline{\hat{x}}_{k+1} = \mathcal{E}\left(\underline{\hat{x}}_k, \underline{u}_k, \underline{y}_k\right)$$



$$\underline{x}_{k+1} = \underline{f}\left(\underline{x}_k, \underline{u}_k\right) \quad \underline{x}_L \text{ is unknown}$$

$$\underline{y}_k = \underline{h}\left(\underline{x}_k\right)$$

Compute $\underline{x}_L$ such that the error between the observation and the model prediction is minimized.

$$\min_{\underline{x}_L} \sum_{k=L}^{K} \left(\underline{\hat{y}}_k - \underline{y}_k\right)^T \underline{R} \left(\underline{\hat{y}}_k - \underline{y}_k\right) \quad \text{(assumption of a perfect model)}$$

Extension:

$$\underline{x}_{k+1} = \underline{f}\left(\underline{x}_k, \underline{u}_k\right) + \underline{w}_k$$

measurements $\quad \underline{y}_{k+1} = \underbrace{\underline{h}\left(\underline{x}_k\right)}_{\text{given}} + \underline{v}_k$

$$\min_{\underset{\text{initial Condition}}{\rightarrow \underline{x}_L}, \underline{w}_L, \underline{w}_{L+1}, \underline{w}_{L+K}, \underline{v}_L, \underline{v}_{L+1} \dots \underline{v}_{L+K}} \sum_{k=L}^{K} \left(\underline{w}_k^T \underline{Q} \underline{w}_k + \underline{v}_k^T \underline{R} \underline{v}_k\right)$$

under the Condition that the model equations are satisfied.

$$\underline{v}_k = \underbrace{\underline{y}_{k+1}}_{\text{measurement}} - \underbrace{\underline{h}\left(\underline{x}_k\right)}_{\text{estimate}}$$

Moving Horizon: performed repetitively for shifted windows

$$\left(t_L \dots t_k\right) \rightarrow \left(t_{L+1} \dots t_{k+1}\right) \rightarrow \dots$$

change of the initial state must be also penalized $\left(\underbrace{x_{L+1|K+1}}_{\text{estimate of } x_{L+1} \text{ at time } k+1} - x_{L+1|K}\right)^T \underline{P} \left(\underline{x}_{L+1|K+1} - x_{L+1|K}\right)$

Advantages: no linearization, easier tuning

Disadvantage: nonlinear optimization must be performed online

Porsche of estimation schemes

5. Model predictive Control

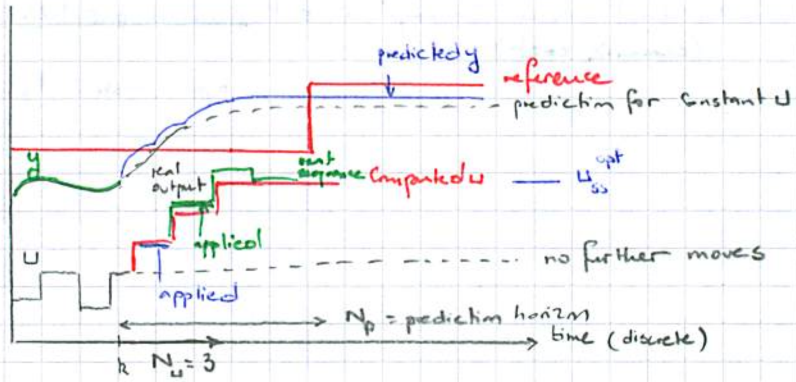
started as a heuristic approach to Control industrial processes

charles Cutler } 1970's
Jacques Richalet }

1) predict the futur behaviour as a function of future inputs

2) optimize the input over a <u>finite</u> horizon

3) Apply only the next move, wait and iterate.



all based on open-loop optimization

The method for multivariable control in the process industries.

$$d_k = y_k^m - y_k^n$$

prediction of the measured value; measurement

usually assumed that the error is constant

Constant offset between model and real plant

→ leads to an integrator in the controller → steady-state accuracy.

DMC Controller is just another linear Controller unless there are constraints.

The key reason for the success of MPC is the ability to handle Constraints on inputs, outputs, and states

<u>General cost function</u>

$$\min J_k = \sum_{j=1}^{N_p} \underline{e}_{k+j}^T \, Q_j \, \underline{e}_{k+j} \qquad \underline{e}_{k+j} = \underline{y}_{k+j}^{ref} - \hat{\underline{y}}_{k+j}$$

$$+ \sum_{j=0}^{N_u} \Delta\underline{u}_{k+j}^T \, \underline{S}_j \, \Delta\underline{u}_{k+j}$$

$$+ \sum_{j=0}^{N_u} \left(\underline{u}_{k+j} - \underline{u}_{ss}^{opt}\right)^T \underline{R}_j \left(\underline{u}_{k+j} - \underline{u}_{ss}^{opt}\right)$$

$\underline{u}_{ss}^{opt}$: optimal steady state input.

while respecting the constraints on $\underline{u}_{k+j}$, $\hat{\underline{y}}_{k+j}$

often $\underline{R}_j = \underline{0}$

except for $j = N_a$

the future of MPC

- nonlinear first principles models
- integration of stationary and dynamic optimization. real time optimization RTO
  - direct optimizing control
  - dynamic real-time optimization

Ex: Minimize (over horizon) solvent consumption (economic cost) s.t (such that) product purities are met traditional control good enters via the constraints.

input constraints are met