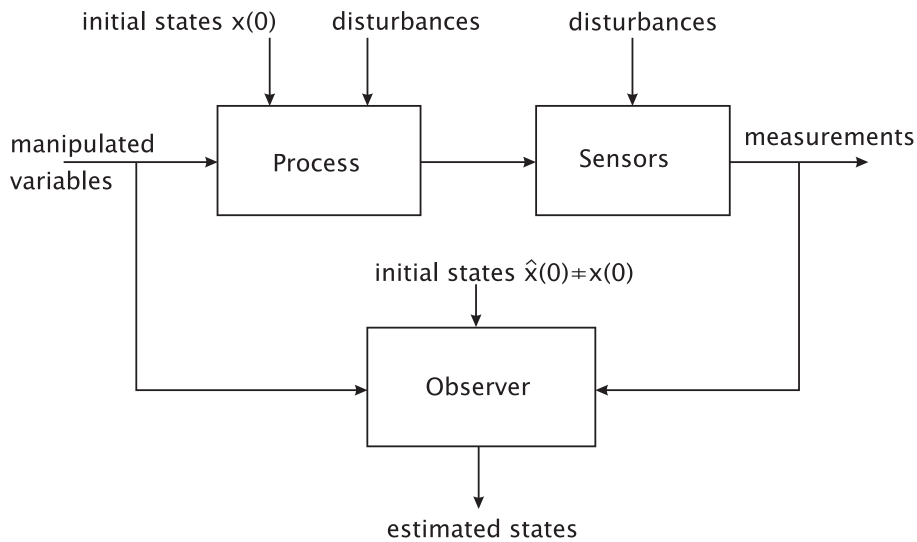

Figure 2.4: Block diagram of a state estimator/observer

The theory on deterministic state observers was developed by Luenberger in the sixties [Lue64] for linear systems. Kalman [KB61] considered stochastic linear systems. In contrast to deterministic observers the concepts for stochastic systems are called state estimators.

These two concepts are discussed in the following for linear systems given by:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{2.45}$$

The observer is described by the system equations, an additive correction term and certain initial conditions. The equations read as:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \\ \hat{\mathbf{x}}(0) &= \hat{\mathbf{x}}_0. \end{aligned} \tag{2.46}$$

The symbol $\hat{}$ determines the estimated/observed states. The different approaches for the observer design differs in the calculation of the matrix $\mathbf{K}$ as shown in the following subsections. From the latter equation it follows that the estimates are correct for true initial conditions $\hat{\mathbf{x}}(t=0) = \mathbf{x}_0$, as the correction term equals zero and stays zero if the dynamic model of the process is exact. This property of an observer is known as simulation property.

Simulation property: For the same initial conditions $\hat{\mathbf{x}}(t=0) = \mathbf{x}(t=0)$ of the observer and the system to be observed it holds $\hat{\mathbf{x}}(t) = \mathbf{x}(t) \; \forall \quad t > 0$.

As usually not all initial conditions are known, especially the unmeasured state variables are not known exactly, a second important property of an observer is the so called convergence property:

Convergence property: If $\hat{\mathbf{x}}(t = 0) \neq \mathbf{x}(t = 0)$, than the estimation error $\tilde{\mathbf{x}}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$ tends to zero for $t \to \infty$.

This property is directly used as a design approach for the Luenberger observer which is discussed in the following subsection.

2.3.1 Luenberger Observer

Following the convergence property, the estimation error $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ must tend to zero for any initial state of the observer. This means that the corresponding differential equation for $\tilde{\mathbf{x}}$ must be stable. Hence, all eigenvalues of this differential equation have to be stable. From (2.45 and 2.46) an autonomous differential equation for the estimation error can be derived:

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{KC})\tilde{\mathbf{x}}, \qquad \tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 - \mathbf{x}_0. \tag{2.47}$$

If the eigenvalues of the matrix $(\mathbf{A} - \mathbf{KC})$ are in the left half complex plane the estimation error decreases exponentially. If the system is observable, the matrix $\mathbf{K}$ can be used to shift the eigenvalues.

Therefore, the design equations for the Luenberger observer read as:

$$\det\left[s\mathbf{I} - (\mathbf{A} - \mathbf{KC})\right] = \prod_{\nu=1}^{n} (s - \lambda_{K,\nu}) \tag{2.48}$$

This pole placement is usually performed by a comparison of the coefficients of the characteristic equations given by (2.48) (see also section 2.1.1).

The Kalman filter discussed in the next section determines $\mathbf{K}$ by the analytical solution of an optimization problem. Matlab provides the `place` command for the pole placement in state feedback control. Due to the duality the same command can be applied for the design of the Luenberger observer.

2.3.2 Kalman Filter

The state space representation of the Kalman Filter(KF) is the same as introduced in equation (2.46), but a different approach for the calculation of the gain matrix $\mathbf{K}$ compared to the Luenberger Observer is applied. In order to derive the equations of the KF stochastic disturbances are introduced into the model equations:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t).\end{aligned} \tag{2.49}$$

The disturbances $\mathbf{w}$ (process noise) and $\mathbf{v}$ (sensor noise) are stationary continuous-time white Gaussian noise with zero mean. Their covariance matrices are defined by:

$$\begin{aligned}\mathrm{cov}[\mathbf{w}(t);\mathbf{w}(\tau)] &= \mathrm{E}\{\mathbf{w}(t)\mathbf{w}^T(\tau)\} = \mathbf{Q}\delta(t-\tau)\\ \mathrm{cov}[\mathbf{v}(t);\mathbf{v}(\tau)] &= \mathrm{E}\{\mathbf{v}(t)\mathbf{v}^T(\tau)\} = \mathbf{S}\delta(t-\tau),\end{aligned} \tag{2.50}$$

where $\mathrm{E}(\cdot)$ denotes the expectation and $\delta(t-\tau)$ is the Dirac delta function (impulse at $t=\tau$) and the covariance matrices $\mathbf{Q}$ and $\mathbf{S}$ are positive definite.

It has to be mentioned that the stochastic behavior assumed in this approach is an artificial one as white noise does not exist in nature, but it is helpful in engineering [Lev95].

The goal of the Kalman Filter is to minimize the expectation of the estimation error $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$. Hence, the cost function to be minimized reads as:

$$\begin{aligned}J &= \mathrm{E}\left\{\sum_{i=1}^{n} \tilde{x}_i^2(t)\right\}\\ &= \mathrm{E}\{\tilde{\mathbf{x}}(t)^T\tilde{\mathbf{x}}(t)\} = \mathrm{tr}[\mathrm{E}\{\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^T(t)\}].\end{aligned} \tag{2.51}$$

Physically interpreted this cost function minimizes the sum of the error variances of the estimation error of each state variable. If $\mathbf{P}$ defines the covariance matrix of the estimation error, i. e.

$$\mathbf{P} = \mathrm{E}\{\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^T(t)\} \tag{2.52}$$

the cost function reads as:

$$J = \mathrm{tr}[\mathbf{P}]. \tag{2.53}$$

The analytical solution of the stated optimization problem can be written as:

$$\mathbf{K} = \mathbf{P}\mathbf{C}\mathbf{S}^{-1} \tag{2.54}$$

and

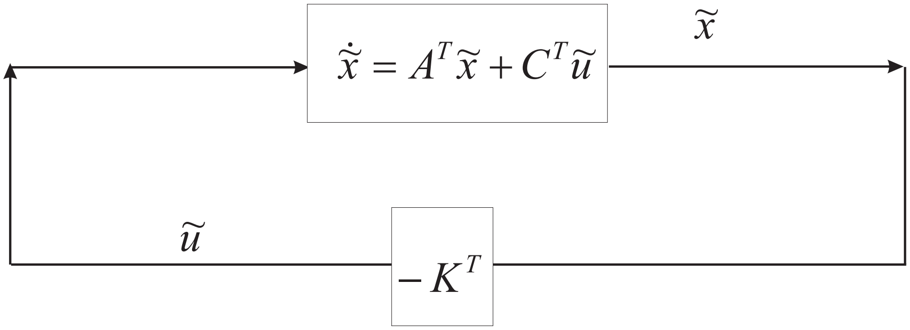

Figure 2.5: Fictitious state feedback controller for the estimation error

$$\mathbf{0} = \mathbf{P}\mathbf{C}^T\mathbf{S}^{-1}\mathbf{C}\mathbf{P} - \mathbf{A}\mathbf{P} - \mathbf{P}\mathbf{A}^T - \mathbf{Q}. \tag{2.55}$$

This solution is similar to the solution derived for the linear quadratic regulator (2.27-2.28). By transposing the results for the controller it follows:

$$\mathbf{0} = \mathbf{P}\mathbf{B}^T\mathbf{Q}_u^{-1}\mathbf{B}\mathbf{P} - \mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P} - \mathbf{Q}_x \tag{2.56}$$

$$\mathbf{R}^T = \mathbf{P}\mathbf{B}\mathbf{Q}_u^{-1}. \tag{2.57}$$

From the upper equations it follows that the solution of the Kalman Filter equals the transposed solution of the linear quadratic regulator. This is known as the duality of controller and observer design and is based on the possibility to represent the estimation problem as a state feedback control problem (see figure 2.5).

The concept of duality is also valid for the controllability and observability matrices. It holds that every controllability result has a corresponding observability result and vice versa [Lev95].

2.3.3 Disturbance/ Parameter Estimation

The disturbances occurring in dynamic systems are usually unknown. Additionally parameter uncertainties are often present in such models. They can also be considered as unknown disturbances. Hence, approaches to the estimation of such disturbances are important for process control. In this section the extension of the concept of state estimation to parameter estimation is discussed.

Even though the exact disturbance is not known, usually an idea of the character is present. This can be used to extend the model equation for the estimator/observer design. In the literature it is often assumed that the disturbances have a zero order hold behavior, i. e. they are stepwise constant. The differential equation to describe this behavior is:

$$\dot{\mathbf{d}} = \mathbf{H}\mathbf{d} \qquad \mathbf{d}(0) = \mathbf{d}_0 \tag{2.58}$$

with $\mathbf{H} = \mathbf{0}$.

But it is also possible to apply sinusodial or ramp signals and the corresponding differential equations.

The zero order hold assumption is valid for disturbances/parameters that are constant or change slowly with time, e. g. the overall heat transfer coefficient in batch polymerization reactors.

The resulting system to be observed reads as:

$$\begin{aligned} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{d}} \end{pmatrix} &= \begin{pmatrix} \mathbf{A} & \mathbf{E} \\ \mathbf{0} & \mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{pmatrix} \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{d} \end{pmatrix} . \end{aligned} \tag{2.59}$$

It is necessary that this extended system is observable. Then the disturbances can be estimated from the available inputs and outputs of the system. It can be shown that it is impossible to estimate more parameters than the number of independent measurements.

Disturbance estimation is useful for deviation control of disturbances but also for monitoring of processes.