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Advanced Process Control - Tutorial 4 **WS 10/11**

State Estimation - Extended Kalman Filter

Exercise - Discrete extended Kalman filter

Consider a continuously stirred tank reactor (CSTR) with a reaction

$$\begin{aligned} A &\overset{k_{AB}}{\Longrightarrow} B \\ 2B &\underset{k_{CB}}{\overset{k_{BC}}{\rightleftharpoons}} C. \end{aligned} \tag{1}$$

Assume that the total volume entering is equal to the volume leaving the system: $\dot{V}_{in} = \dot{V}_{out} = \dot{V} = const$. The input of the system is the concentration of component A at the reactor inlet (c_{A0}) , the measured outputs are the concentrations of component B (c_B) and C (c_C). The component balances are:

$$\frac{d}{dt}c_A = \frac{\dot{V}}{V_R}(c_{A0} - c_A) - k_{AB}c_A \tag{2}$$

$$\frac{d}{dt}c_B = -\frac{\dot{V}}{V_R}c_B + k_{AB}c_A + k_{CB}c_C - k_{BC}{c_B}^2 \tag{3}$$

$$\frac{d}{dt}c_C = -\frac{\dot{V}}{V_R}c_C + k_{BC}{c_B}^2 - k_{CB}c_C \tag{4}$$

where all parameters other than k_{AB} are known. In this tutorial, a discrete Extended Kalman Filter (EKF) shall be designed to observe the unmeasurable concentrations A and the unknown parameter k_{AB}. It is assumed that the initial conditions are:

$$\mathbf{x}_0 = [c_A \quad c_B \quad c_C \quad k_{AB}]_0^T = [0.1 \quad 0.1 \quad 0.1 \quad 0.1]^T$$

The covariance matrix R is known:

$$\mathbf{R} = \mathrm{diag}([0.01^2,\ 0.01^2])$$

The system is fully observable in the region of attainable states.

Tasks: Download the zip-file Tutorial_5, unpack it and store the files in one folder (e.g. Tutorial_5).

1. How does the measurement matrix look like?

2. Transform the system equations to a discrete-time system by applying the finite difference approach:

$$\frac{d\,x(t)}{dt} \approx \frac{x_k - x_{k-1}}{t_k - t_{k-1}} = \frac{x_k - x_{k-1}}{\Delta t}$$

   where Δt refers to the integration step width.

3. Implement the discretized model in the m-file `model_discret`, which will act as your observer model in the following.

$$x_k = f(x_{k-1}, \Delta t)$$

4. The discretized model is used to predict the system states, however an EKF needs to correct the states a linearized version of the model at the the sampling points. Please derive the Jacobian of the discrete model with respect to the states and implement it in `model_discret` as well.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_{k-1}}$$

5. Open the m-file `A4_Nonlinear_System_Euler.m`. Add the measurement matrix and tune your observer by choosing suitable tuning parameters $P0$ and Q. Please do not change the sampling time and integration step width.

6. Increase the integration step width to 1. What can be observed? What must been changed in order to get back to a better performance?

7. Now open the m-file `A5_Nonlinear_System_Continuous.m`. In this simulation file, the observer uses the continuous equation between two measurements. Apply the best tuning of your observer you got so far. How is the performance of this observer?

Discrete Kalman filter:
Consider the following nonlinear dynamic system:

$$\dot{x} = f(x, u) + w(t) \tag{5}$$
$$x(0) = x_0 + w(0) \tag{6}$$
$$y(t) = h(x) + v(t) \tag{7}$$

w and v are assumed to be

- mean-free: $E(w) = E(v) = 0$
- uncorrelated: $E(w_i, w_j) = E(v_i, v_j) = 0 \quad \forall i \neq j$ and
- distributed according to a normal distribution with covariance matrices Q $(w(t))$, R $v(t))$ and P_0 $(w(0)))$.

The time continuous system can be transformed into a discrete form by integration between two sample times:

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} (f(x(t), u(t)) + w(t))dt \quad k = 0, 1, \cdots \tag{8}$$
$$:= F(\hat{x}_k, u_k) + w_k \tag{9}$$
$$y_k = h(x_k) + v(k) \tag{10}$$

Instead of the continuous Matrix-Ricatti-Equation (see lecture notes) an algebraic Matrix-Riccati-Equation for the covariance matrix of the estimation error can be derived which can easily be solved. Therefore, the algorithm of the discrete EKF can be devided into two steps:

1. Correction or Measurement update

$$K_k = P_{k,k-1}\, C_{k,k-1}^T \left(C_{k,k-1} P_{k,k-1} C_{k,k-1}^T + R\right)^{-1} \tag{11}$$
$$\hat{x}_{k,k} = \hat{x}_{k,k-1} + K_k \left(y_k - h(\hat{x}_{k,k-1})\right) \tag{12}$$
$$P_{k,k} = \left(I - K_k C_{k,k-1}\right) P_{k,k-1} \tag{13}$$

2. Prediction or Time update
   Based on the last corrected estimate (filtered state) $\hat{x}_{k,k}$, the states for the next step are predicted by the model equations without considering disturbances:

$$\hat{x}_{k+1,k} = F(\hat{x}_{k,k}, u_k) \tag{14}$$

   Furthermore, the covariance matrix of the estimation error is predicted for the next sample time:

$$P_{k+1,k} = A_{k,k} P_{k,k} A_{k,k}^T + Q \tag{15}$$

   with:

$$A_{k,k} = \left.\frac{\partial F}{\partial x}\right|_{\hat{x}_{k,k}, u_k} \qquad C_{k,k-1} = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k,k}} \tag{16}$$

The variable $\hat{x}_{k,k-1}$ represents the estimated state at time t_k, that is calculated from available measurements up to time t_{k-1}.

Notations:

Q can be interpreted as a weighting factor for the model accuracy. Decreasing the values of the elements of Q yield a larger weight of the model equations, i.e. the model is assumed to be more accurat.

R can be interpreted as a weighting factor for the reliability of the measurement. Decreasing the values of the elements of R yield in an increasing weight of the concerning measurements, i.e. the measurement is assumed to be more reliable.

The covariance matrix of the initial error P_0 has a large influence on the initial convergence behaviour of the EKF and can be interpreted as a weighting factor of the initial guess, i.e. small values of the elements of P_0 imply a high accuracy of estimate.