

State Estimation - Dealing with noise and model errors

Solutions

Quickies

1. No, "unobservable" only means that the error dynamics of the observer cannot be chosen freely. The unobservable eigenvalues " λ_u " of the system will be eigenvalues of the observer system.

λ_u stable $\forall u \rightarrow$ error converges to zero but not arbitrarily fast ($\propto \exp(\lambda_u t)$)

2. Because of the separation principle: Feedback dynamics and observer dynamics can be designed independently !

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} A & BK \\ LC & A + BK - LC \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}.$$

Change of coordinates: $\hat{x} = x + e \rightarrow$

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BK & BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}.$$

In the new coordinates the system has block triangular form, thus the eigenvalues are on the main diagonale \rightarrow Set of closed loop eigenvalues = Set of eigenvalues designed by state feedback \oplus Set of eigenvalues designed by error dynamics

3. The observer eigenvalues should be placed such that its dynamic is faster than the one of the dominant eigenvalues of A , respectively $A + BK$ in a closed loop. However, the observer should not be too fast because of measurement noise as this may lead to huge under- and overshoots of the observer system. Like for controller design, a rule of thumb is to make the error eigenvalues 10 times faster than the system eigenvalues.

4. For a Kalman Filter it is assumed that the system is disturbed by stochastic noise (system noise $v(t)$ and measurement noise $w(t)$). Both are white Gaussian with zero mean:

$$\begin{aligned}\dot{x} &= Ax + Bu + v(t), x(0) = x_0 \\ y &= Cx + w(t)\end{aligned}\quad (1)$$

The Kalman Filter equations are the solution of the minimization problem of the expected error between plant and observer:

$$\min E \left[(x - \hat{x})^T (x - \hat{x}) \right]$$

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ L &= PC^T R^{-1} \quad \text{error feedback gain} \\ \dot{P} &= AP + PA^T + Q - PC^T R^{-1} CP \quad \text{Riccati Differential Equation}\end{aligned}\quad (2)$$

with Q as the covariance matrix of v , R as the covariance matrix of w and P as the covariance matrix of $(x - \hat{x})$.

5. $P(0)$, Q , R . R can often be taken from the measurement devices. When $P(0)$ is small, you trust your initial values; when Q/R is small you trust your model and when R/Q is small you trust your measurements.

Exercise - State Estimation

- Using the rule of thumb and making the error dynamics ten times faster than the system dynamics lead to heavy oscillations around the real state values (cf. Figure 1). The measurement noise is simply amplified by the observer gain. However, by analyzing the system eigenvalues one can achieve a much better tuning. The eigenvalues and corresponding eigenvectors are:

D =

$$\begin{bmatrix} -0.1000 & 0 & 0 \\ 0 & -5.1000 & 0 \\ 0 & 0 & -1.6000 \end{bmatrix}$$

V =

$$\begin{bmatrix} 0 & 0 & 0.7548 \\ 0.5547 & 0.7071 & -0.1078 \\ 0.8321 & -0.7071 & -0.6470 \end{bmatrix}$$

The eigenvalues -5.1 is already quite fast in comparison to the other two and hence there is no need to change it. The dominating eigenvalue is -0.1 , so this should be made faster. Changing the third eigenvalue -1.6 should be avoided as it is hard to move from the point of view of component B (see the eigenvector). Changing this eigenvalue lead directly to a high gain (e.g. double it and look at the gain), which results in a bad noise performance as B is the noisy component. A good tuning is $[-5.1 \ -1 \ -1.6]$ (cf. Figure 2). Tuning the Kalman Filter here is equivalent to choosing seven free parameters being R , q_{11} , q_{22} , q_{33} , and p_{011} , p_{022} , p_{033} . R can be found by analyzing the measurement device (here: Band-Limited White Noise block in Simulink). To minimize the number of parameters further, one usually choose P_0 to be somehow depended on Q , e.g. $Q/100$ or $Q * 10$. Then, you only have to play with the diagonal entries of Q . A good tuning is:

$$R = 0.01, \quad Q = \text{diag}([0.001, 0.00001, 0.001]), \quad P_0 = Q * 200.$$

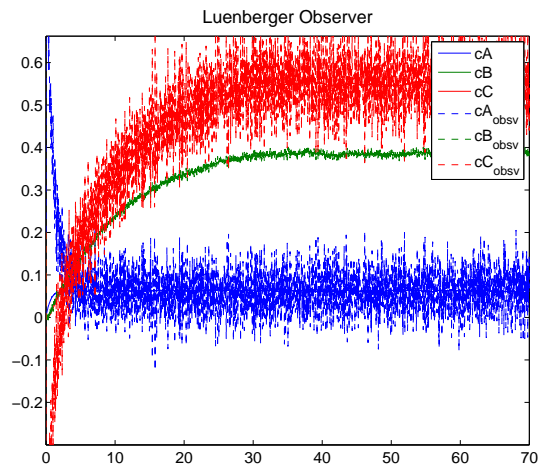


Figure 1: Luenberger Observer, observer ten times faster than system

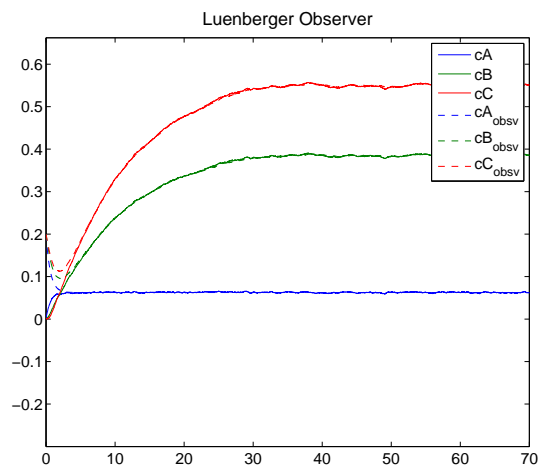


Figure 2: Luenberger Observer, eigenvalues at $[-5.1 \ -1 \ -1.6]$

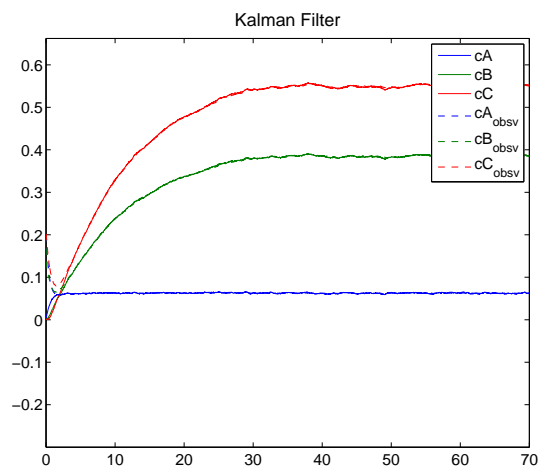


Figure 3: Kalman Filter

2. For almost all observer designs there will be a steady state offset between observer and plant because of the model error:

$$\begin{aligned}\dot{\hat{x}} &= \hat{A}\hat{x} + Bu + LC(x - \hat{x}) && \text{Observer with wrong A matrix} \\ \dot{x} &= Ax + Bu && \text{Plant} \\ \dot{e} &= \hat{A}\hat{x} - A\hat{x} + LC(x - \hat{x}) && \text{Error}\end{aligned}\quad (3)$$

By splitting the wrong system matrix into the correct matrix A and an error (disturbance) E , $\hat{A} = A + E$, it follows:

$$\begin{aligned}\dot{e} &= A\hat{x} - A\hat{x} + LC(x - \hat{x}) + E\hat{x} \\ &= (A - LC)e + E\hat{x} \\ &= (\hat{A} - LC)e + Ex\end{aligned}\quad (4)$$

Thus the steady state of the error can be written as:

$$e = -(\hat{A} - LC)^{-1} Ex.$$

Taking the above mentioned tuning parameter values, the following results shown in Figure 4 are obtained. Similar results can be derived for other values. An option

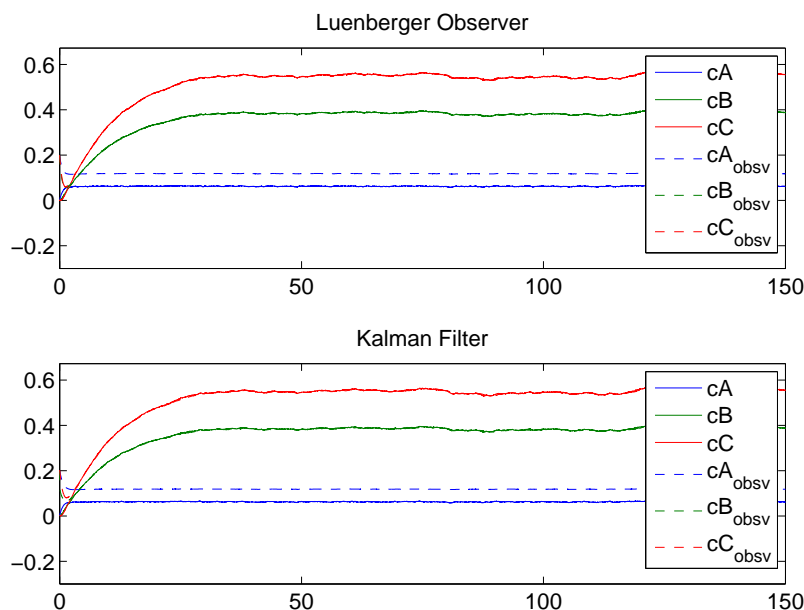


Figure 4: Estimation with plant model mismatch

to achieve steady state convergence is to find a suitable observer gain that fulfill the relation:

$$(\hat{A} - LC)^{-1} E = 0 \quad (5)$$

This can be fulfilled by decoupling the disturbance from the steady state error. To do this, one first has to spend a few thoughts on the disturbance to estimate how it

might look like. In this case it is obvious that the model of component A is wrong (cf. Figure 4). If we assume that the input is known correctly, the model error must be part of k_{ABC_A} . Hence we can model the disturbance as:

$$E = \begin{bmatrix} -\Delta k_{AB} & 0 & 0 \\ \Delta k_{AB} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this it follows that the model error can be decoupled from the observer by choosing the first column of $(\hat{A} - LC)^{-1}$ equals to the second column and to solve these three equations for the three unknown entries of L . This results in:

$$L_3 = \frac{\hat{A}_{32} [\hat{A}_{33} (\hat{A}_{21} + \hat{A}_{11}) - \hat{A}_{31} (\hat{A}_{23} + \hat{A}_{13})]}{\hat{A}_{33} (\hat{A}_{11} + \hat{A}_{21}) - \hat{A}_{31} (\hat{A}_{23} + \hat{A}_{13})} \quad (6)$$

$$L_2 = 10 \quad \text{arbitrarily}$$

$$L_1 = \hat{A}_{22} + \hat{A}_{12} - L_2 - \frac{(\hat{A}_{32} - L_3) (\hat{A}_{23} + \hat{A}_{13})}{\hat{A}_{33}}$$

Tuning the Luenberger observer in this way one obtains:

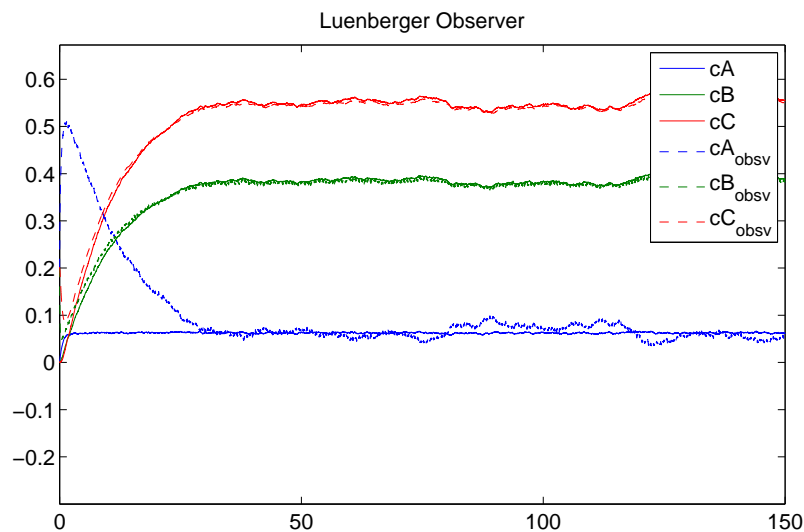


Figure 5: Observer with a steady state error decoupled from the disturbance