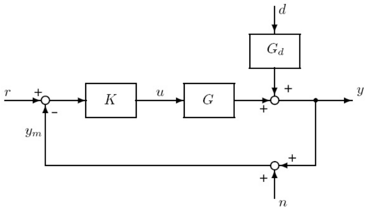
FEL3210 Multivariable Feedback Control

Elling W. Jacobsen
Automatic Control Lab, KTH

Lecture 2: Performance Limitations in SISO Systems (Ch. 5)

Last time



Control error

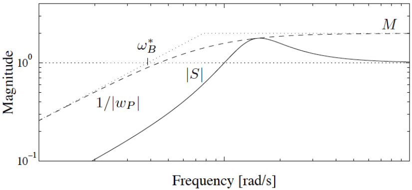
$$e = -Sr + SG_d d + Tn$$

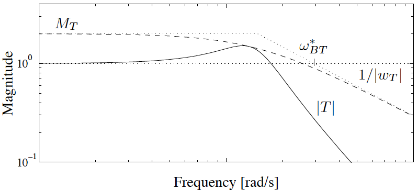
Aim: design controller K so that

- $|S(j\omega)|$ is small for frequencies where d and r important
- $|T(j\omega)|$ is small for frequencies where n large

approaches: loop shaping, signal based optimization, ...

Shaping S and T





$$|S| < 1/|w_p| \; \forall\omega \;\Leftrightarrow\; \|w_P S\|_\infty < 1$$

$$|T| < 1/|w_T| \; \forall\omega \;\Leftrightarrow\; \|w_T T\|_\infty < 1$$

- **Q:** can we shape S and T freely, i.e., choose any weights w_P, w_T?
- **A:** No! there exist a number of fundamental constraints
  - *algebraic constraints*
  - *analytic constraints*

  and also *practical constraints* such as bounds on the manipulated variables

Outline

- Algebraic constraints
  - $S + T = 1$
  - Interpolation constraints
- Analytic constraints
  - preliminaries from analytic function theory
  - RHP poles and zeros
  - Bode Sensitivity Integral and extensions
- Practical constraints: input constraints
- Summary: a procedure for controllability analysis
- (Exercise 1)

Algebraic constraint I: S+T=1

Recall

$$S = \frac{1}{1+L}\ ; \quad T = \frac{L}{1+L}$$

Hence

$$S(j\omega) + T(j\omega) = 1 \quad \forall\omega$$

It follows that, at any frequency

- $|S(j\omega)| > 0.5$ or $|T(j\omega)| > 0.5$
  - cannot deal effectively with both disturbances and measurement noise at the same frequency
  - cannot choose $|w_P| > 1$ and $|w_T| > 1$ at the same frequency
- $|S| >> 1 \;\Leftrightarrow\; |T| >> 1$
  - amplifying disturbances implies amplification also of noise and vice versa

Algebraic constraint II: interpolation constraints

$$S(s) = \frac{1}{1+L(s)} \; ; \quad T(s) = \frac{L(s)}{1+L(s)} \qquad L(s) = G(s)K(s)$$

- Let z denote a RHP zero of $G(s)$ or $K(s)$. Then

$$S(z) = 1 \; ; \quad T(z) = 0$$

  - follows since internal stability implies that $L(s)$ must have the same RHP zero, i.e., $L(z) = 0$

- Let p denote a RHP pole of $G(s)$ or $K(s)$. Then

$$S(p) = 0 \; ; \quad T(p) = 1$$

  - as above, $L(p) = \infty$ due to requirement of internal stability

Algebraic constraint II: interpolation constraints

$$S(s) = \frac{1}{1+L(s)}\ ; \quad T(s) = \frac{L(s)}{1+L(s)} \qquad L(s) = G(s)K(s)$$

- Let z denote a RHP zero of $G(s)$ or $K(s)$. Then

$$S(z) = 1\ ; \quad T(z) = 0$$

  - follows since internal stability implies that $L(s)$ must have the same RHP zero, i.e., $L(z) = 0$

- Let p denote a RHP pole of $G(s)$ or $K(s)$. Then

$$S(p) = 0\ ; \quad T(p) = 1$$

  - as above, $L(p) = \infty$ due to requirement of internal stability

Preliminaries I: The Maximum Modulus Theorem

Maximum Modulus Thm. *Suppose that* Ω *is a region in the complex plane and* F *is an analytic function in* Ω *and, furthermore, that* F *is not equal to a constant. Then* $|F|$ *attains its maximum value at the boundary of* Ω.

- S and T are stable transfer-functions and hence analytic in the complex RHP, for which the boundary is the $j\omega$-axis.
- A trivial consequence is

$$\|S\|_\infty \geq S(z) = 1 \; ; \quad \|T\|_\infty \geq T(p) = 1$$

  However, not too useful bounds. Need to add weights to get meaningful constraints.

Lower Bound on Weighted Sensitivity from RHP zero

- From Maximum Modulus Thm, with RHP zero z

$$\|w_P S\|_\infty \geq |w_P(z)S(z)| = |w_P(z)|$$

- Thus, since control objective is $\|w_P S\|_\infty < 1$ we require

$$|w_P(z)| < 1$$

- Example: consider weight

$$w_P(s) = \frac{s/M + \omega_B^*}{s}$$

  - if $M = \infty$, then $w_P(z) = \omega_B^*/z$ and $\omega_B^* < z$
  - if $M = 2$, then $w_P(z) = (z/2 + \omega_B^*)/z$ and $\boxed{\omega_B^* < z/2}$

Lower Bound on Weighted Complimentary Sensitivity from RHP pole

- From Maximum Modulus Thm, with RHP pole p

$$\|w_T T\|_\infty \geq |w_T(p)T(p)| = |w_T(p)|$$

- Thus, with control objective $\|w_T T\|_\infty < 1$ we require

$$|w_T(p)| < 1$$

- Example: consider weight

$$w_T(s) = \frac{M_T s + \omega_{BT}^*}{\omega_{BT}^* M_T}$$

  - if $M_T = \infty$, then $w_P(z) = p/\omega_{BT}^*$ and $\omega_{BT}^* > p$
  - if $M_T = 2$, then $w_T(p) = (2p + \omega_{BT}^*)/2\omega_{BT}^*$ and $\boxed{\omega_{BT}^* > 2p}$

KTH Electrical Engineering

Combined RHP pole and RHP zero - bandwidth limitations

Assume $\omega_B \approx \omega_{BT} \approx \omega_c$ and we require $M < 2, M_T < 2$. Then,

- for a RHP zero

$$\omega_c < z/2$$

- for a RHP pole

$$\omega_c > 2p$$

Thus, can only achieve acceptable performance if $2p < z/2$ or

$$z > 4p$$

– poles and zeros close to eachother in the RHP are fundamentally difficult to deal with

KTH Electrical Engineering

Combined RHP pole and RHP zero - minimum peaks

- Recall that $S(p) = 0$. Factor sensitivity function S as

$$S = S_{mp} \underbrace{\frac{s-p}{s+p}}_{S_{ap}}$$

- It follows that, since $S(z) = 1$,

$$S_{mp}(z) = S_{ap}^{-1}(z) = \frac{z+p}{z-p}$$

- Maximum Modulus Thm

$$\|w_P S\|_\infty = \|w_P S_{mp}\|_\infty \geq |w_P(z) S_{mp}(z)| = |w_P(z) \frac{z+p}{z-p}|$$

- Example: $w_P = 1$

$$\|S\|_\infty \geq \frac{|z+p|}{|z-p|}$$

- Similarily, $T(z) = 0$ and we get

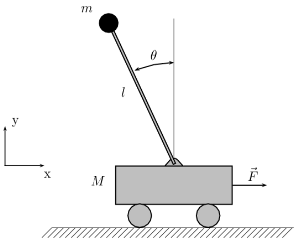
$$\|w_T T\|_\infty \geq |w_T(p) \frac{p+z}{p-z}|$$

- Example: $w_T = 1$

$$\|T\|_\infty \geq \frac{|z+p|}{|z-p|}$$

Thus, combination of RHP pole and RHP zero greatly amplifies the effect they would have alone!

Example: Stabilization of Cart-Pendulum



$$X(s) = \frac{ls^2 - g}{s^2(Mls^2 - (M+m)g)} F(s)$$

$$z = \sqrt{\frac{g}{l}}, \quad p = z\sqrt{1 + m/M}$$

- With $l = 1$ and $m = M$: $z = \sqrt{10},\ p = \sqrt{20} \quad \Rightarrow$

$$\|S\|_\infty > 5.8, \|T\|_\infty > 5.8$$

- With $l = 1$ and $m = 0.1M$: $z = \sqrt{10},\ p = \sqrt{11} \quad \Rightarrow$

$$\|S\|_\infty > 42, \|T\|_\infty > 42$$

RHP poles and control limitations

- RHP poles combined with other bandwidth limitations, such as time delays and input constraints, give similar results
- Example: 1st order Padé approximation of time-delay

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s} \quad \Rightarrow \quad z = \frac{2}{\theta}$$

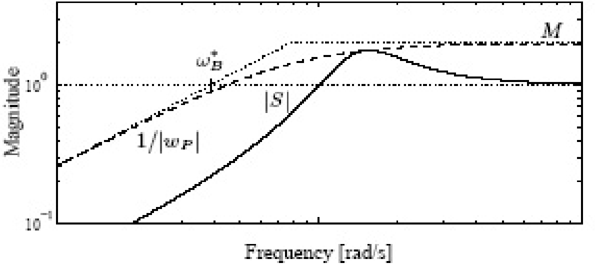
- Real life examples:

X-29

JAS 39 Gripen

Fundamental trade-off between different frequencies

Plot of typical sensitivity function

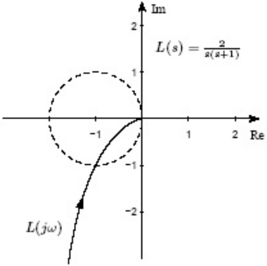


- frequencies where $|S| < 1$: feedback improves performance
- frequencies where $|S| > 1$: feedback deteriorates performance

Shall see: decreasing $|S|$ at one frequency has to be compensated by increasing $|S|$ at some other frequency.

Pole excess of two $\rightarrow$ $\max_\omega |S| > 1$

Assume loop-gain $L(s)$ is stable and has pole excess ≥ 2, then the distance between $L(j\omega)$ and -1 is less than 1 for some ω



$$|1 + L(j\omega)| < 1 \quad \Leftrightarrow \quad |S(j\omega)| > 1$$

"Proof": $\arg L(j\omega)$ will be between $-\pi/2$ and $-\pi$, i.e., L passes 3rd quadrant, for some frequencies, and $|L| \rightarrow 0$ and $\arg L \leq -\pi$ as $\omega \rightarrow \infty$. Finally, for closed-loop stability L may no encircle -1.

Preliminaries II: Cauchy Integral Theorem

Cauchy's Thm. Suppose that Ω is an open, simply connected set and Γ is a non-self-intersecting closed contour in Ω, Then, if F is an analytic function in Ω

$$\int_\Gamma F(s)ds = 0$$

Alternative formulation

- Let $\gamma : [0, 1] \to \Omega$ be a differentiable function such that $\gamma(0) = \gamma(1)$
- Then

$$\int_0^1 F(\gamma(t))\gamma'(t)dt = \int_\Gamma F(s)ds = 0$$

The Sensitivity Integral - open-loop stable systems

Assume $L(s)$ is stable and rational with relative degree $n_r > 1$. Then, for closed-loop stability, the sensitivity function $S(s) = (1 + L(s))^{-1}$ must satisfy

$$\int_0^\infty \ln |S(j\omega)| d\omega = 0$$

- note: consider $\ln |S|$ versus linear ω-axis
- area for $|S| < 1$ must be exactly matched by area for $|S| > 1$
- *"Waterbed effect":* pushing down sensitivity at one frequency increases sensitivity at some other frequency

Sketch of proof

- The function $\ln|S(s)|$ is analytic on the RHP, hence

$$\int_D \ln|S(s)|ds = \int_{C_i} \ln|S(s)|ds + \int_{C_\infty} \ln|S(s)|ds = 0$$

i.e.,

$$j\int_0^\infty \ln|S(j\omega)|d\omega = \frac{1}{2}\int_{C_\infty} \ln|1+L(s)|ds$$

- For large s, $\ln|1+L(s)| \approx \ln|1+as^{-n_r}| \approx |as^{-n_r}|$, so on C_E with $\gamma = Ee^{jt}$

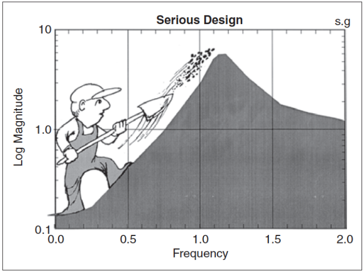
$$\begin{aligned}\frac{1}{2}\ln|1+L(s)| \quad &\approx \quad \int_0^{\pi/2} |\frac{a}{E^{n_r}}e^{-jn_r t}|Eje^{jt}dt = \\ &-\frac{aj}{E^{n_r-1}}\int_0^{\pi/2} e^{jt}dt = -\frac{aj}{E^{n_r-1}}\frac{\pi}{2}\end{aligned}$$

- For $n_r > 1$ the integral converges to zero which gives the result.

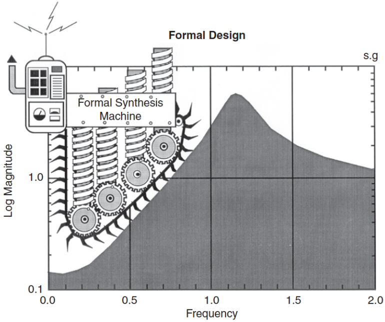
Making the trade-off

From Stein (IEE CS, 2003, see homepage):

Manual loop-shaping:



Optimization:



Bode Sensitivity Integral

Theorem 5.1: *assume $L(s)$ rational with relative degree $n_r > 1$ and with N_P RHP poles at p_i. Then, for closed-loop stability, the sensitivity function must satisfy*

$$\int_0^{\infty} \ln |S(j\omega)| d\omega = \pi \sum_{i=1}^{N_P} Re(p_i)$$

- Proof sketch: write $\hat{S}(s) = S(s) \prod_i \frac{s+p_i}{s-p_i}$ which yields integral as above but with addition of

$$\sum_{i=1}^{N_P} \int_D \ln \frac{|s+p_i|}{|s-p_i|} ds = -j\pi \sum_{i=1}^{N_P} p_i$$

KTH Electrical Engineering

Sensitivity Integral - RHP zeros

- The Bode Sensitivity Integral applicable to all systems
- When $L(s)$ has a RHP zero z, the sensitivity function must also satisfy the integral (Freudenberg and Looze, 1988)

$$\int_0^\infty \ln|S(j\omega)| \cdot w(z,\omega) d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{\bar{p}_i - z} \right|$$

where

$$w(z,\omega) = \frac{2z}{z^2 + \omega^2}$$

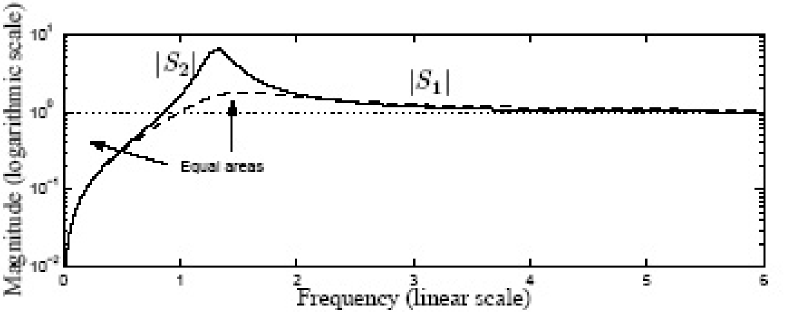
– the weight $w(z,\omega)$ falls off with a -2 slope from $\omega = z$, i.e., effectively cuts of contributions for frequencies $\omega > z$, i.e., for a stable system

$$\int_0^z \ln|S| d\omega \approx 0$$

Trade-off must be made over a limited frequency range

Sensitivity function for systems with loop-gains

$$L_1 = \frac{2}{s(s+1)} \; ; \quad L_2 = L_1 \frac{-s+5}{s+5}$$



Summary Fundamental Constraints

- For all systems: $S + T = 1 \ \forall\omega$
- if RHP zero at $s = z$ then $\|w_P S\|_\infty < 1$ require $|w_P(z)| < 1$
  - e.g., $M_S < 2 \quad \Rightarrow \quad \omega_B < z/2$
- if RHP pole at $s = p$ then $\|w_T T\|_\infty <$ require $|w_T(p)| < 1$
  - e.g., $M_T < 2 \quad \Rightarrow \quad w_{BT} > 2p$
- Sensitivity reduction at on frequency must always be traded against a sensitivity increase at another frequency

$$\int_0^\infty \ln |S| d\omega = 0$$

Summary cont'd

- Combined RHP pole and RHP zero can impose much more severe constraints than individual effects, e.g.,

$$\|w_P S\|_\infty \geq |w_P(z)\frac{z+p}{z-p}| \; ; \quad \|w_T T\|_\infty \geq |w_T(z)\frac{p+z}{p-z}|$$

$$\|S\|_\infty \geq \frac{|z+p|}{|z-p|} \; ; \quad \|T\|_\infty \geq \frac{|z+p|}{|z-p|}$$

$$\int_0^\infty \ln|S(j\omega)| \cdot w(z,\omega)d\omega = \pi \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{\bar{p}_i - z} \right|$$

- For similar limitations on other closed-loop transfer-functions see S&P Table 5.1

Practical Limitations: Input Constraints

Input for perfect control $e = 0$

$$u = G^{-1} r - G^{-1} G_d d$$

- **Disturbances:** $r = 0$ and $|d| = 1$ yields

$$|u| = |G^{-1} G_d| < 1 \quad \forall \omega$$

  corresponds to requiring $|G| > |G_d| \; \forall \omega$.
- **Setpoints:** $d = 0$ and $|r| = R$ yields

$$|u| = |G^{-1} R| < 1 \quad \forall \omega < \omega_r$$

  corresponds to requiring $|G| > R \; \forall \omega < \omega_r$

For *acceptable control*, i.e., $|e| < 1$, requirements are relaxed to

$$|G| > |G_d| - 1 \quad \forall \omega \; ; \quad |G| > |R| - 1 \quad \forall \omega$$

Practical Limitations: Input Constraints

Input for perfect control $e = 0$

$$u = G^{-1}r - G^{-1}G_d d$$

- **Disturbances:** $r = 0$ and $|d| = 1$ yields

$$|u| = |G^{-1}G_d| < 1 \quad \forall\omega$$

  corresponds to requiring $|G| > |G_d| \; \forall\omega$.

- **Setpoints:** $d = 0$ and $|r| = R$ yields

$$|u| = |G^{-1}R| < 1 \quad \forall\omega < \omega_r$$

  corresponds to requiring $|G| > R \; \forall\omega < \omega_r$

For *acceptable control*, i.e., $|e| < 1$, requirements are relaxed to

$$|G| > |G_d| - 1 \quad \forall\omega \; ; \quad |G| > |R| - 1 \quad \forall\omega$$

Practical Limitations: Input Constraints

Input for perfect control $e = 0$

$$u = G^{-1}r - G^{-1}G_d d$$

- **Disturbances:** $r = 0$ and $|d| = 1$ yields

  $$|u| = |G^{-1}G_d| < 1 \quad \forall\omega$$

  corresponds to requiring $|G| > |G_d| \; \forall\omega$.
- **Setpoints:** $d = 0$ and $|r| = R$ yields

  $$|u| = |G^{-1}R| < 1 \quad \forall\omega < \omega_r$$

  corresponds to requiring $|G| > R \; \forall\omega < \omega_r$

For *acceptable control*, i.e., $|e| < 1$, requirements are relaxed to

$$|G| > |G_d| - 1 \quad \forall\omega \; ; \quad |G| > |R| - 1 \quad \forall\omega$$

KTH Electrical Engineering

Practical Limitations: Input Constraints

Input for perfect control $e = 0$

$$u = G^{-1} r - G^{-1} G_d d$$

- **Disturbances:** $r = 0$ and $|d| = 1$ yields

  $$|u| = |G^{-1} G_d| < 1 \quad \forall \omega$$

  corresponds to requiring $|G| > |G_d| \; \forall \omega$.
- **Setpoints:** $d = 0$ and $|r| = R$ yields

  $$|u| = |G^{-1} R| < 1 \quad \forall \omega < \omega_r$$

  corresponds to requiring $|G| > R \; \forall \omega < \omega_r$

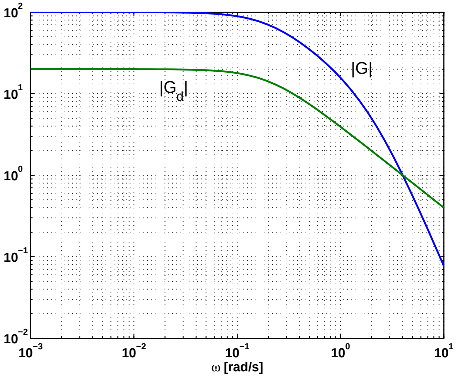
For *acceptable control*, i.e., $|e| < 1$, requirements are relaxed to

$$|G| > |G_d| - 1 \quad \forall \omega \; ; \quad |G| > |R| - 1 \quad \forall \omega$$

Example

From exercise 1:

$$y = \frac{100}{(5s+1)(0.5s+1)^2}\, u \; + \; \frac{20}{5s+1}\, d$$



$|G| > |G_d| - 1$ at all frequencies, but close to limit.

Summary: a simple procedure for controllability analysis

Assume system has been scaled as described above

- **Performance requirements from disturbances / setpoints:** require $|SG_d| < 1$ or $|S| < 1/|G_d|$. Corresponds to bandwidth requirement

$$\omega_B > \omega_d$$

Similar for setpoints, require $|S| < 1/|R|$ up to $\omega = \omega_r$

- **Requirement from RHP poles:** RHP pole at p yields requirement

$$\omega_{BT} > 2p$$

- **Limitations from RHP zeros:** RHP zero at z yields approximate limitation

$$\omega_B < z/2$$

Summary: a simple procedure for controllability analysis

Assume system has been scaled as described above

- **Performance requirements from disturbances / setpoints:** require $|SG_d| < 1$ or $|S| < 1/|G_d|$. Corresponds to bandwidth requirement

$$\omega_B > \omega_d$$

  Similar for setpoints, require $|S| < 1/|R|$ up to $\omega = \omega_r$

- **Requirement from RHP poles:** RHP pole at p yields requirement

$$\omega_{BT} > 2p$$

- **Limitations from RHP zeros:** RHP zero at z yields approximate limitation

$$\omega_B < z/2$$

Summary: a simple procedure for controllability analysis

Assume system has been scaled as described above

- **Performance requirements from disturbances / setpoints:** require $|SG_d| < 1$ or $|S| < 1/|G_d|$. Corresponds to bandwidth requirement

  $$\omega_B > \omega_d$$

  Similar for setpoints, require $|S| < 1/|R|$ up to $\omega = \omega_r$
- **Requirement from RHP poles:** RHP pole at p yields requirement

  $$\omega_{BT} > 2p$$

- **Limitations from RHP zeros:** RHP zero at z yields approximate limitation

  $$\omega_B < z/2$$

KTH Electrical Engineering

Summary: a simple procedure for controllability analysis

Assume system has been scaled as described above

- **Performance requirements from disturbances / setpoints:** require $|SG_d| < 1$ or $|S| < 1/|G_d|$. Corresponds to bandwidth requirement

$$\omega_B > \omega_d$$

  Similar for setpoints, require $|S| < 1/|R|$ up to $\omega = \omega_r$
- **Requirement from RHP poles:** RHP pole at p yields requirement

$$\omega_{BT} > 2p$$

- **Limitations from RHP zeros:** RHP zero at z yields approximate limitation

$$\omega_B < z/2$$

- **Limitations from time delay:** time delay θ yields

$$\omega_B < 1/\theta$$

- **Limitations from input constraints:** require

$$|G| > |G_d| - 1 \; ; \quad |G| > |R| - 1$$

If any conflicts between requirements and limitations, then modify requirements or redesign your system!

- **Limitations from time delay:** time delay θ yields

$$\omega_B < 1/\theta$$

- **Limitations from input constraints:** require

$$|G| > |G_d| - 1 \; ; \quad |G| > |R| - 1$$

If any conflicts between requirements and limitations, then modify requirements or redesign your system!

- **Limitations from time delay:** time delay θ yields

$$\omega_B < 1/\theta$$

- **Limitations from input constraints:** require

$$|G| > |G_d| - 1 \; ; \quad |G| > |R| - 1$$

If any conflicts between requirements and limitations, then modify requirements or redesign your system!

Exercise 1 – one solution

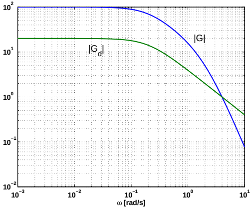
Unscaled system

$$\hat{G}(s) = \frac{5}{(5s+1)(0.5s+1)^2}\ ; \quad \hat{G}_d(s) = \frac{2}{5s+1}$$

- Scaling: $|y| < 0.1 = D_y$, $|d| < 1 = D_d$, $|u| < 2 = D_u$, $R = ?$
  $G = D_y^{-1}\hat{G}D_u\ ; \quad G_d = D_y^{-1}\hat{G}_d D_d$

$$G(s) = \frac{100}{(5s+1)(0.5s+1)^2}\ ; \quad G_d(s) = \frac{2}{5s+1}$$

- Controllability: main limitation is input constraint

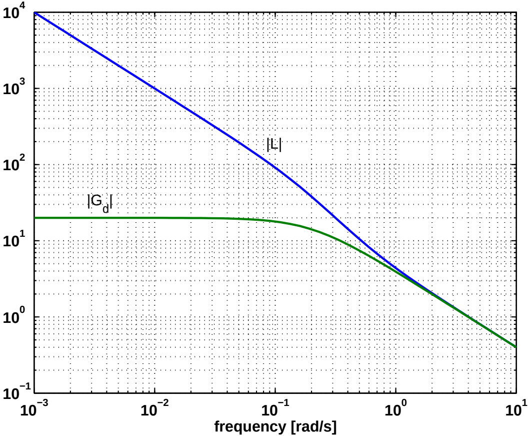


Specifications should be feasible, but relatively tight

- Bandwidth requirements:
  - for disturbances: $\omega_B \approx 4$
  - for setpoints: $\omega_B \approx 2$ (rise time 1 for $\tau \approx 0.5$)

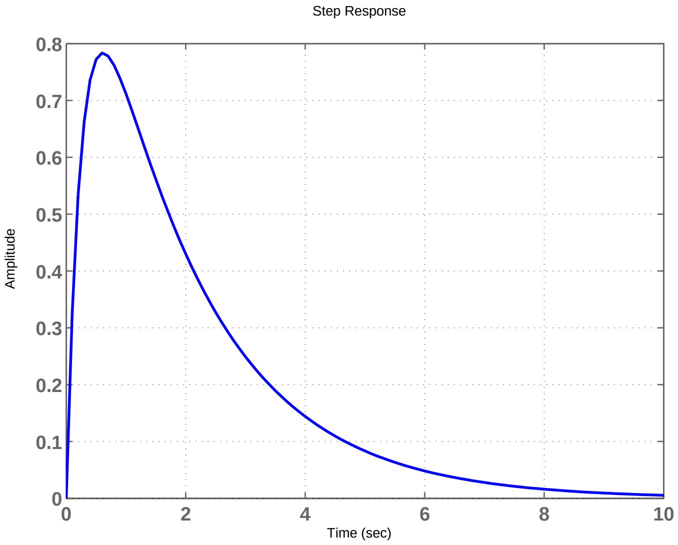
- Design for disturbance: try first with loop-gain

$$L = \frac{s + w_I}{s} G_d$$



$w_I = 0.5$ gives acceptable disturbance response

Step response:



OK!

Inverse based design gives improper controller

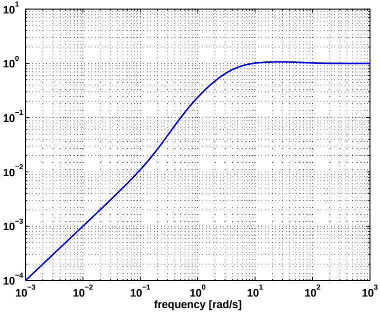
$$K_1 = \frac{s + w_I}{s} \frac{(s+2)^2}{5}$$

make proper by adding poles at high frequency

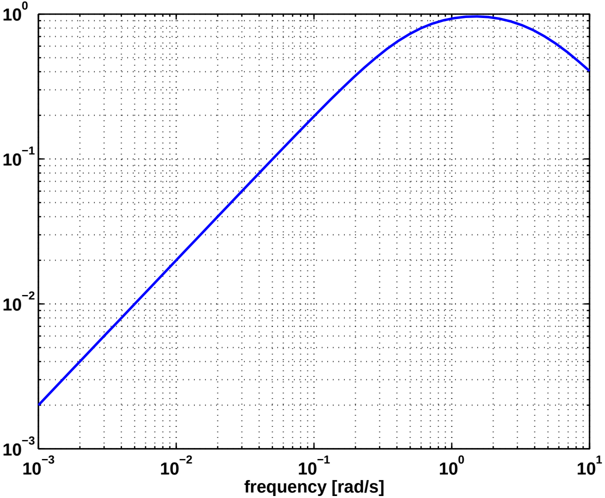
$$K = K_1 \frac{1}{(0.01s+1)^2}$$

Get essentially same response

Plot of $|S|$

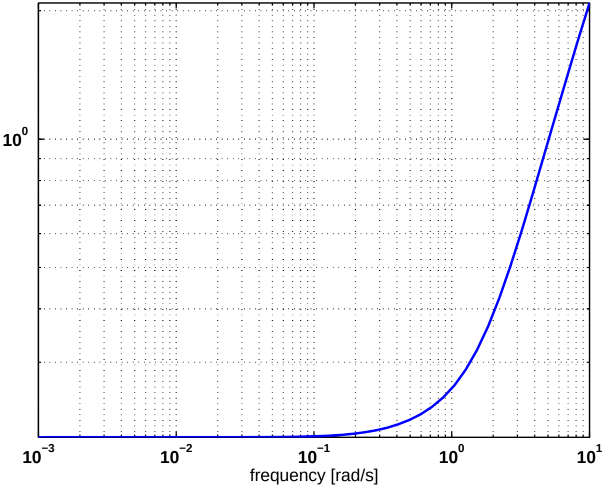


Plot of $|SG_d|$:



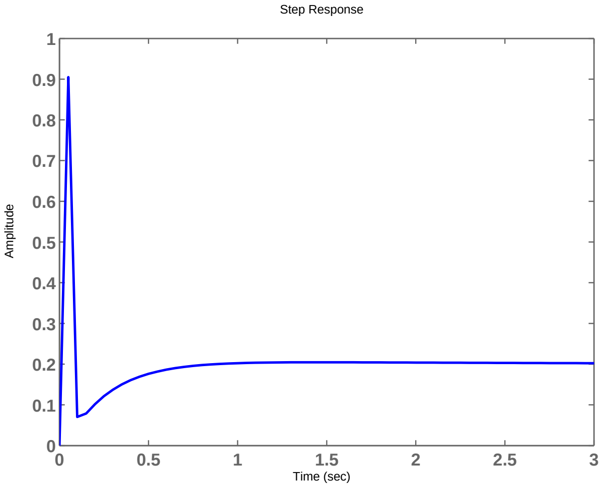
- Input usage

$$|u| = |KSG_d|$$



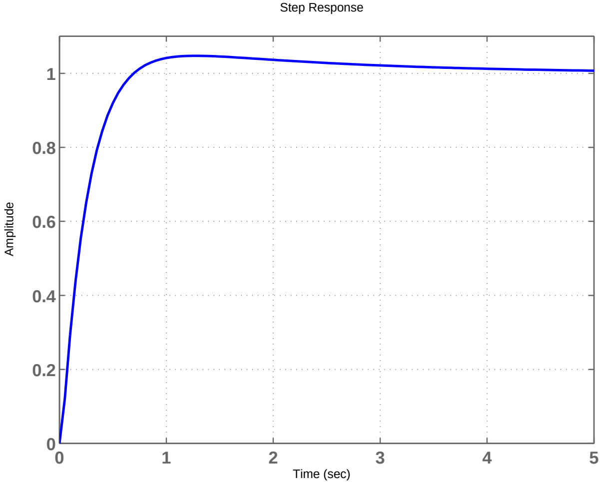
Apparent problems at high frequencies, may need to add furher filtering poles in K at high frequencies.

Input for step disturbance:



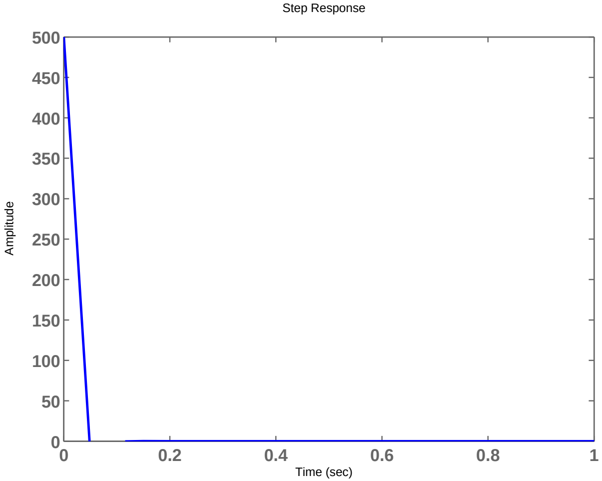
OK!

Bandwidth should be OK for setpoints. Step response for unit step in reference:



OK!

Corresponding input:



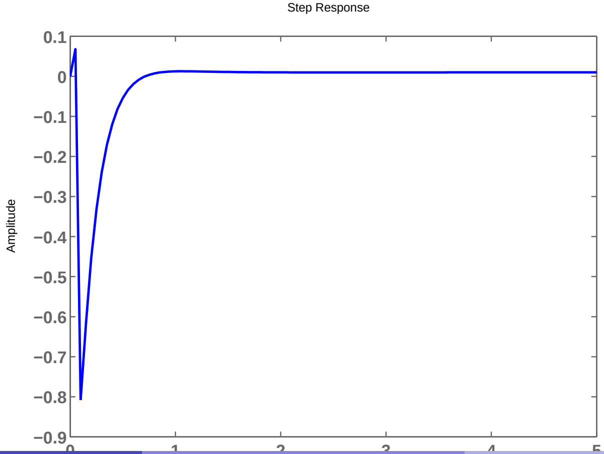
Unacceptable!

Solution: add prefilter on setpoint (2-DOF controller)

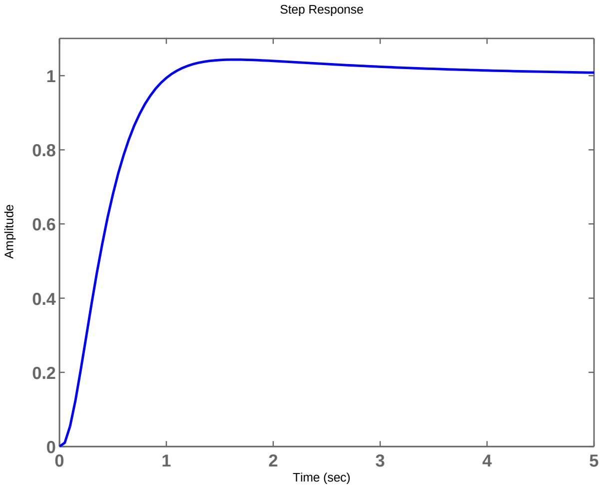
$$F_r = \frac{1}{0.2s + 1}$$

gives sufficient damping of $|KS|$ at high-frequencies

u:



response in y:



OK!