ME 450 – Multivariable Robust Control

Linear Quadratic Regulator

Continuous Dynamic Optimization

1. Distinctions between continuous and discrete systems

1- Continuous control laws are simpler

2- We must distinguish between *differentials* and *variations* in a quantity

2. The calculus of variations

If $x(t)$ is a continuous function of time t, then the differentials $dx(t)$ and dt are not independent. We can however define a small change in $x(t)$ that is independent of dt. We define the *variation* $\delta x(t)$, as the incremental change in $x(t)$ when time t is held fixed.

What is the relationship between $dx(t)$, dt, and $\delta x(t)$?

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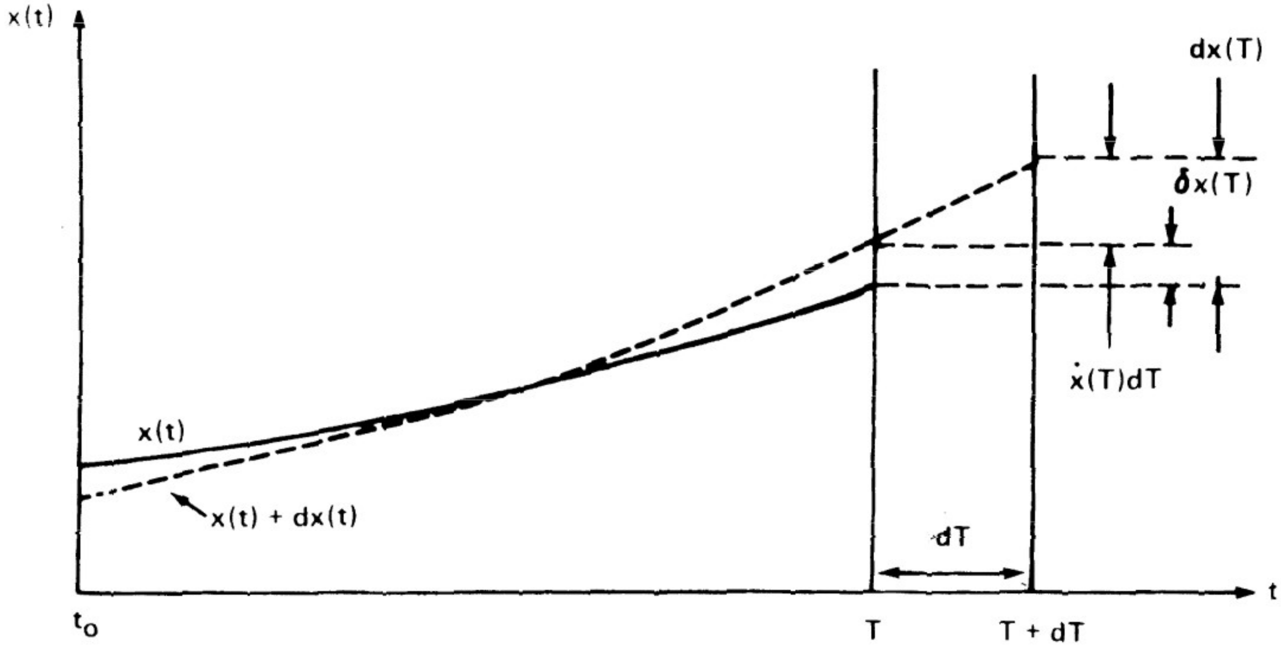
Final time variation: $dx(T) = \delta x(T) + \dot{x}(T)dT$



Leibniz's rule: $J(x) = \int_{t_0}^{T} h\big(x(t),t\big)dt$

$$dJ = h\big(x(T),T\big)dT - h\big(x(t_0),t_0\big)dt_0 + \int_{t_0}^{T}\Big[h_x^T\big(x(t),t\big)\delta x\Big]dt$$

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3. Continuous Dynamic Optimization

The plant is described by the general nonlinear continuous-time time-varying dynamical equation

$$\dot{x} = f(t, x, u), \qquad t_0 < t < T$$

with initial condition x_0 given. The vector x has n components and the vector u has m components.

The problem is to find the sequence $u^*(t)$ on the time interval $[t_0, T]$ that drives the plant along a trajectory $x^*(t)$, minimizes the performance index

$$J(t_0) = \phi(T, x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\,dt$$

and such that

$$\psi(T, x(T)) = 0$$

Continuous Dynamic Optimization

We adjoin the constraints (system equations and terminal constraint) to the performance index J with a multiplier function $\lambda(t) \in R^n$ and a multiplier constant $\nu \in R^p$.

$$\bar{J}(t_0) = \phi(T, x(T)) + \nu^T \psi(T, x(T)) + \int_{t_0}^{T} \left[L(t, x(t), u(t)) + \lambda^T(t) \left(f(t, x, u) - \dot{x} \right) \right] dt$$

For convenience, we define the *Hamiltonian* function

$$H(t, x, u) = L(t, x, u) + \lambda^T(t) f(t, x, u)$$

Thus,

$$\bar{J}(t_0) = \phi(T, x(T)) + \nu^T \psi(T, x(T)) + \int_{t_0}^{T} \left[H(t, x(t), u(t), \lambda(t)) - \lambda^T(t) \dot{x} \right] dt$$

Continuous Dynamic Optimization

We want to examine now the increment in $\bar{J}$ due to increments in all the variables x, λ, ν, u and t. Using Leibniz's rule, we compute

$$d\bar{J}(t_0) = \left(\phi_x + \psi_x^T \nu\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu\right)dt\big|_T + \psi^T\big|_T d\nu$$

$$+\left(H - \lambda^T \dot{x}\right)dt\big|_T - \left(H - \lambda^T \dot{x}\right)dt\big|_{t_0}$$

$$+\int_{t_0}^{T}\left[H_x^T \delta x + H_u^T \delta u - \lambda^T \delta\dot{x} + \left(H_\lambda - \dot{x}\right)^T \delta\lambda\right]dt$$

We integrate by parts, $\int_{t_0}^{T}\lambda^T \delta\dot{x}\,dt = \lambda^T \delta x\big|_T - \lambda^T \delta x\big|_{t_0} - \int_{t_0}^{T}\dot{\lambda}^T \delta x\,dt$, to obtain

$$d\bar{J}(t_0) = \left(\phi_x + \psi_x^T \nu - \lambda^T\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu + H - \lambda^T \dot{x} + \lambda^T \dot{x}\right)dt\big|_T$$

$$+\psi^T\big|_T d\nu - \left(H - \lambda^T \dot{x} + \lambda^T \dot{x}\right)dt\big|_{t_0} + \lambda^T dx\big|_{t_0}$$

$$+\int_{t_0}^{T}\left[\left(H_x + \dot{\lambda}\right)^T \delta x + H_u^T \delta u + \left(H_\lambda - \dot{x}\right)^T \delta\lambda\right]dt$$

$$\boxed{dx(t) = \delta x(t) + \dot{x}(t)dt}$$

Continuous Dynamic Optimization

We assume that t_0 and $x(t_0)$ are both fixed and given, then dt_0 and $dx(t_0)$ are both zero. According to the Lagrange theory the constrained minimum of J is attained at the unconstrained minimum of $\bar{J}$. This is achieved when $d\bar{J} = 0$ for all independent increments in its arguments. Then, the necessary conditions for a minimum are:

$$\psi\big|_T = 0$$

$$\left.\begin{aligned} &H_\lambda - \dot{x} = 0 \Rightarrow \dot{x} = H_\lambda = f \\ &H_x + \dot{\lambda} = 0 \Rightarrow -\dot{\lambda} = H_x = L_x + \lambda^T f_x \end{aligned}\right\} \text{Two-point Boundary-value Problem}$$

$$H_u = L_u + \lambda^T f_u = 0$$

$$\left(\phi_x + \psi_x^T \nu - \lambda^T\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu + H\right)^T dt\big|_T = 0$$

The initial condition for the Two-point Boundary-value Problem is the known value for x_0. For a fixed T, the final condition is either a desired value of $x(T)$ or the value of $\lambda\ (T)$ given by the last equation. This equation allows for possible variations in the final time $T \rightarrow$ minimum time problems.

Continuous Dynamic Optimization

System Properties | SUMMARY | Controller Properties

System Model

$$\dot{x}(t) = f(t, x, u)$$

Performance Index

$$J(t_0) = \phi(T, x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt$$

Final-state Constraint

$$\psi(T, x(T)) = 0$$

Hamiltonian

$$H(t, x, u, \lambda) = L(t, x, u) + \lambda(t) f(t, x, u)$$

State Equation

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f(t, x, u), \qquad t \geq t_0$$

Costate Equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}, \qquad t \leq T$$

Stationary Condition

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0$$

Boundary Condition

$$x(t_0) \ \text{given}$$

$$\left(\phi_x + \psi_x^T \nu - \lambda^T\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu + H\right)^T dt\big|_T = 0$$

Continuous Dynamic Optimization

4. Linear Quadratic Regulator (LQR) Problem

The plant is described by the linear continuous-time dynamical equation

$$\dot{x} = A(t)x + B(t)u,$$

with initial condition x_0 given. We assume that the final time T is fixed and given, and that no function of the final state ψ is specified. We want to find the sequence $u^*(t)$ that minimizes the performance index:

$$J(t_0) = \frac{1}{2}x^T(T)S(T)x(T) + \frac{1}{2}\int_{t_0}^{T}\left(x^T Q(t)x + u^T R(t)u\right)dt$$

Linear because of the system dynamics

Quadratic because of the performance index

Regulator because of the absence of a tracking objective---we are interested in regulation around the zero state.

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We adjoin the system equations (constraints) to the performance index J with a multiplier sequence $\lambda(t) \in R^n$.

$$\bar{J}(t_0) = \frac{1}{2} x^T(T) S(T) x(T) + \frac{1}{2} \int_{t_0}^{T} \left[x^T Q(t) x + u^T R(t) u \dot{x} + \lambda^T \left(A(t) x + B(t) u - \dot{x} \right) \right] dt$$

We define the Hamiltonian

$$H(t) = x^T Q(t) x + u^T R(t) u \dot{x} + \lambda^T \left(A(t) x + B(t) u \right)$$

Thus, the necessary conditions for a stationary point are:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = A(t) x + B(t) u$$ *State Equation*

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = Q x + A^T(t) \lambda$$ *Costate Equation*

$$\frac{\partial H}{\partial u} = R u + B^T \lambda = 0 \Rightarrow \boxed{u(t) = -R^{-1} B^T \lambda(t)}$$ *Stationary Condition*

Continuous Dynamic Optimization

We must solve the Two-point Boundary-value Problem

$$\dot{x} = \frac{\partial H}{\partial \lambda} = A(t)x - B(t)R^{-1}B^T(t)\lambda(t)$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = Qx + A^T(t)\lambda$$

for $t_0 \le t \le T$, with boundary conditions

$$x(t_0) = x_0$$

We will solve this system for two special cases:

1- Fixed final state $\rightarrow$ Open loop control

2- Free final state $\rightarrow$ Closed loop control

Continuous Dynamic Optimization

4.1 Fixed-Final State and Open-Loop Control

$$\dot{x} = A(t)x + B(t)u, \qquad x(T) = r_T$$

$$J(t_0) = \frac{1}{2}\int_{t_0}^{T} u^T R(t) u\, dt$$

If $Q \neq 0$, the problem is intractable analytically. The Two-point Boundary-value Problem is now simplified:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} = Ax - BR^{-1}B^T\lambda \\ -\dot{\lambda} &= \frac{\partial H}{\partial x} = Qx + A^T\lambda \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} = Ax - BR^{-1}B^T\lambda \\ \dot{\lambda} &= \frac{\partial H}{\partial x} = -A^T\lambda \end{aligned}$$

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The costate equation is decoupled from the state equation, and it has an easy solution:

$$\dot{\lambda} = -A^T \lambda \Rightarrow \boxed{\lambda(t) = e^{A^T(T-t)}\lambda(T)}$$

We replace λ in the state equation and solve:

$$\dot{x} = Ax - BR^{-1}B^T e^{A^T(T-t)}\lambda(T) \Rightarrow \boxed{x(t) = e^{A(t-t_0)}x_0 - \int_{t_0}^{t} e^{A(T-\tau)}BR^{-1}B^T e^{A^T(T-\tau)}\lambda(T)d\tau}$$

We solve now for $\lambda(T)$:

$$x(T) = e^{A(T-t_0)}x_0 - \int_{t_0}^{T} e^{A(T-\tau)}BR^{-1}B^T e^{A^T(T-\tau)}d\tau\lambda(T) = r_T$$

$$\lambda(T) = -G_C^{-1}(t_0,T)\left(r_T - e^{A(T-t_0)}x_0\right) \qquad G_C(t_0,T) = \int_{t_0}^{T} e^{A(T-\tau)}BR^{-1}B^T e^{A^T(T-\tau)}d\tau$$

Weighted Controllability Gramian of $[A,B]$

Continuous Dynamic Optimization

Summary:

$$G_C\left(t_0,T\right)=\int_{t_0}^{T} e^{A(T-\tau)}BR^{-1}B^T e^{A^T(T-\tau)}d\tau$$

The inverse of the gramian $G_C(t_0,T)$ exits if and only if the system is controllable.

$$\lambda\left(T\right)=-G_C^{-1}\left(t_0,T\right)\left(r_T-e^{A\left(T-t_0\right)}x_0\right)$$

$$x\left(t\right)=e^{A\left(t-t_0\right)}x_0-\int_{t_0}^{t} e^{A(T-\tau)}BR^{-1}B^T e^{A^T(T-\tau)}\lambda\left(T\right)d\tau$$

$$u^*\left(t\right)=R^{-1}B^T e^{A^T(T-t)}G_C^{-1}\left(t_0,T\right)\left(r_T-e^{A\left(T-t_0\right)}x_0\right)$$

Continuous Dynamic Optimization

4.2 Free-Final-State and Closed-Loop Control

$$\dot{x} = A(t)x + B(t)u, J(t_0) = \frac{1}{2}x^T(T)S(T)x(T) + \frac{1}{2}\int_{t_0}^{T}\left(x^T Q(t)x + u^T R(t)u\right)dt$$

The Two-point Boundary-value Problem is:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax - BR^{-1}B^T\lambda$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = Qx + A^T\lambda$$

We need $\left[\left.\frac{\partial \phi}{\partial x}\right|_T - \lambda^T(T)\right]^T \left.dx\right|_T = 0 \Rightarrow \lambda^T(T) = \left.\frac{\partial \phi}{\partial x}\right|_T = x^T(T)S(T)$

Let us assume that this relationship holds for all $t_0 \leq t \leq T$ (Sweep Method)

$$\lambda(t) = S(t)x(t)$$

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We differentiate the costate and use the state equation,

$$\dot{\lambda} = \dot{S}x + S\dot{x} = \dot{S}x + S\left(Ax - BR^{-1}B^{T}Sx\right)$$

We use now the costate equation,

$$-\left(Qx + A^{T}Sx\right) = \dot{S}x + S\left(Ax - BR^{-1}B^{T}Sx\right)$$

$$-\dot{S}x = \left(A^{T}S + SA - SBR^{-1}B^{T}S + Q\right)x$$

Since this must hold for any trajectory x,

$$-\dot{S} = A^{T}S + SA - SBR^{-1}B^{T}S + Q$$ Ricatti Equation (RE)

The optimal control is given by,

$$u(t) = -R^{-1}B^{T}Sx(t) = -K(t)x(t)$$ Feedback Control!!!

$$K(t) = R^{-1}B^{T}S(t)$$ Kalman Gain

Continuous Dynamic Optimization

This expresses u as a time-varying, linear, state-variable, feedback control. The feedback gain K is computed ahead of time via S, which is obtained by solving the Riccati equation backward in time with terminal condition S_T.

Similarly to the discrete-time case, it is possible to rewrite the cost function as

$$J(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0) + \frac{1}{2} \int_{t_0}^{T} \left\| R^{-1} B^T S x + u \right\|_R^2 dt$$

If we select the optimal control, the value of the cost function for $t_0 \leq t \leq T$ is just

$$J(t_0) = \frac{1}{2} x^T(t_0) S(t_0) x(t_0)$$

Steady-State Feedback

5. Steady-State Feedback for continuous-time systems

The solution of the LQR optimal control problem for continuous-time systems is a state feedback of the form

$$u(t) = -K(t)x(t)$$

where

$$K(t) = R^{-1}B^T S(t)$$

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q$$

The closed-loop system is time-varying!!!

$$\dot{x}(t) = \big(A - BK(t)\big)x(t)$$

What about a suboptimal constant feedback gain?

$$u(t) = -K(t)x(t) = -K_\infty x(t)$$

Steady-State Feedback

5.1 The Algebraic Riccati Equation (ARE)

$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q \qquad \text{RDE}$$

Let us assume that when $t \rightarrow -\infty$, the sequence $S(t)$ converges to a steady-state matrix S_∞. If $S(t)$ does converge, then $dS/dt = 0$. Thus, in the limit

$$0 = A^T S + SA - SBR^{-1}B^T S + Q \qquad \text{ARE}$$

The limiting solution S_∞ is clearly a solution of the ARE. Under some circumstances we may be able to use the following time-invariant feedback control instead of the optimal control,

$$u = -K_\infty x$$

$$K_\infty = R^{-1}B^T S_\infty$$

Steady-State Feedback

1- When does there exist a bounded limiting solution S_∞ to the Ricatti equation for all choices of $S(T)$?

2- In general, the limiting solution S_∞ depends on the boundary condition $S(T)$. When is S_∞ the same for all choices of $S(T)$?

3- When is the closed-loop system (u=-$K_\infty x$) asymptotically stable?

Theorem: Let (A, B) be stabilizable. Then, for every choice of $S(T)$ there is a bounded solution S_∞ to the RDE. Furthermore, S_∞ is a positive semidefinite solution to the ARE.

Theorem: Let C be a square root of the intermediate-state weighting matrix Q, so that Q=$C^TC\geq 0$, and suppose $R>0$. Suppose (A, C) is observable. Then, (A, B) is stabilizable if and only if:

a- There is a unique positive definite limiting solution S_∞ to the RDE. Furthermore, S_∞ is the unique positive definite solution to the ARE.

b- The closed-loop plant

$$\dot{x} = \left(A - BK_\infty\right)x$$

is asymptotically stable, where K_∞ is given by $K_\infty = R^{-1}B^TS_\infty$