Lecture 7

Chapter 9: Controller Design

Eugenio Schuster

`schuster@lehigh.edu`

Mechanical Engineering and Mechanics

Lehigh University

Controller Design

7 Controller Design [9]

7.1 Trade-offs in MIMO feedback design [9.1]

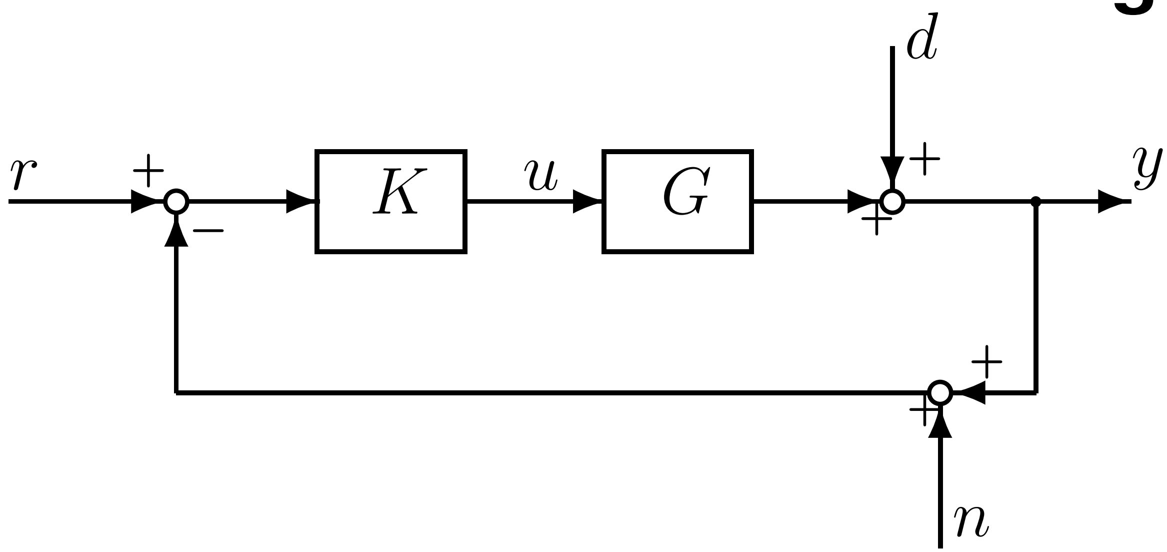

Figure 1: One degree-of-freedom feedback

$$y(s) = T(s)r(s) + S(s)d(s) - T(s)n(s) \tag{7.1}$$

$$u(s) = K(s)S(s)\left[r(s) - n(s) - d(s)\right] \tag{7.2}$$

Controller Design

Closed-loop objectives:

1. For *disturbance rejection* make $\bar{\sigma}(S)$ small.
2. For *noise attenuation* make $\bar{\sigma}(T)$ small.
3. For *reference tracking* make $\bar{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$.
4. For *control energy reduction* make $\bar{\sigma}(KS)$ small.
5. For *robust stability* in the presence of an additive perturbation make $\bar{\sigma}(KS)$ small.
6. For *robust stability* in the presence of a multiplicative output perturbation make $\bar{\sigma}(T)$ small.

The closed-loop requirements 1 to 6 cannot all be satisfied simultaneously. Feedback design is therefore a trade-off over frequency of conflicting objectives.

Controller Design

(7.3) $$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1$$

- At frequencies where $\underline{\sigma}(L) >> 1$, we have $\bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$
- At frequencies where $\bar{\sigma}(L) << 1$, we have $\bar{\sigma}(T) \approx \bar{\sigma}(L)$
- At the bandwidth frequency ($1/\bar{\sigma}(S(j\omega_B)) = \sqrt{2} = 1/41$), we have $0.41 \leq \underline{\sigma}(L(j\omega_B)) \leq 2.41$

Controller Design

Over specified frequency ranges, we can approximate the closed-loop requirements by the following open-loop objectives:

1. For *disturbance rejection* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.
2. For *noise attenuation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
3. For *reference tracking* make $\underline{\sigma}(GK)$ large; valid for frequencies at which $\underline{\sigma}(GK) \gg 1$.
4. For *control energy reduction* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
5. For *robust stability to an additive perturbation* make $\bar{\sigma}(K)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.
6. For *robust stability to a multiplicative output perturbation* make $\bar{\sigma}(GK)$ small; valid for frequencies at which $\bar{\sigma}(GK) \ll 1$.

Controller Design

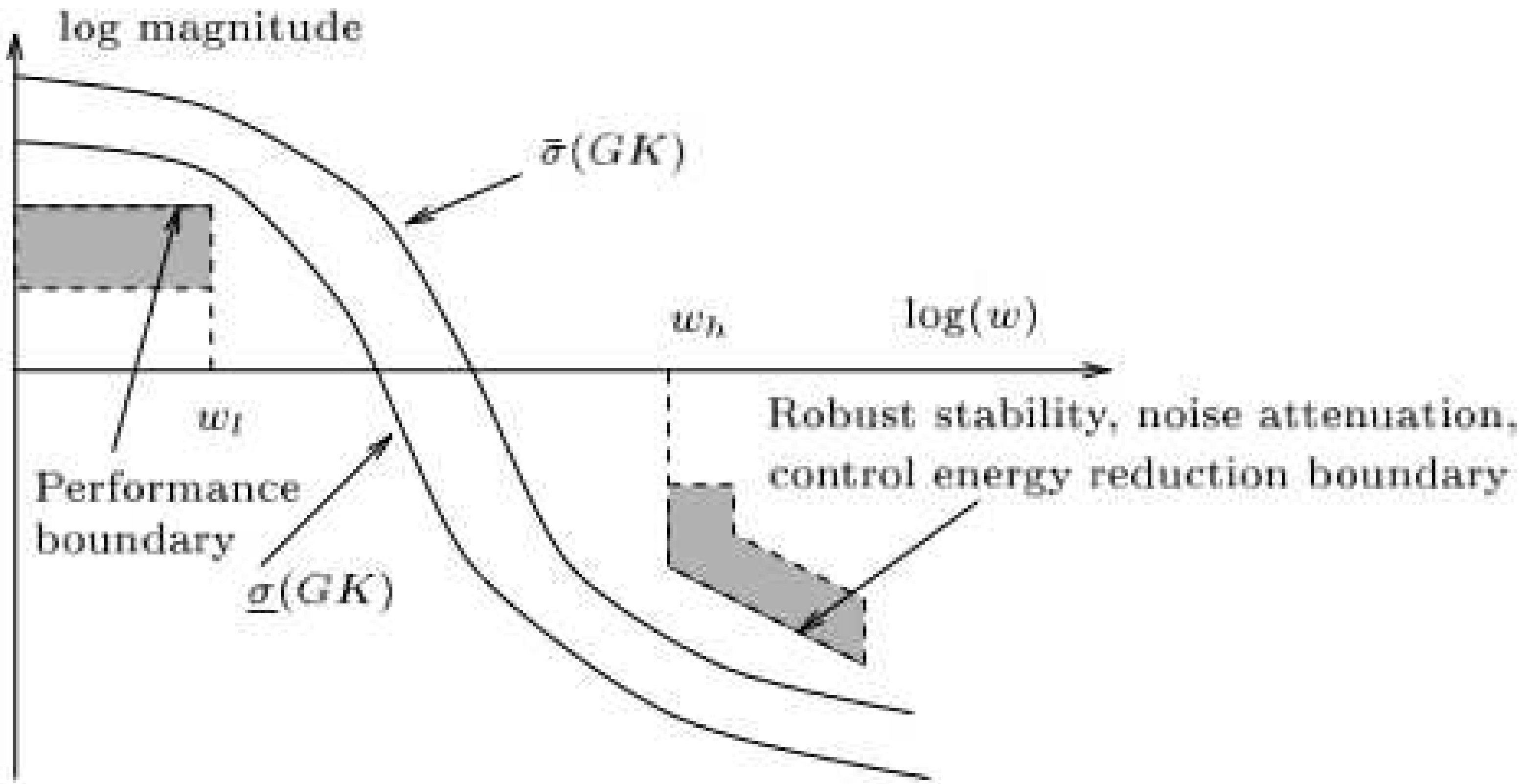

Figure 2: Design tradeoff.

Controller Design

Requirements 1 and 3 are valid and important at low frequencies, $0 \leq \omega \leq \omega_l \leq \omega_B$. Requirements 2, 4, 5 and 6 are conditions which are valid and important at high frequencies, $\omega_B \leq \omega_h \leq \omega \leq \infty$.

At frequencies where we want high gains (at low frequencies) the “worst-case" direction is related to $\underline{\sigma}(L)$, whereas at frequencies where we want low gains (at high frequencies) the “worst-case" direction is related to $\bar{\sigma}(L)$.

Controller Design

7.3 LQG control [9.2]

7.3.1 Traditional LQR and LQG Problems [9.2.1]

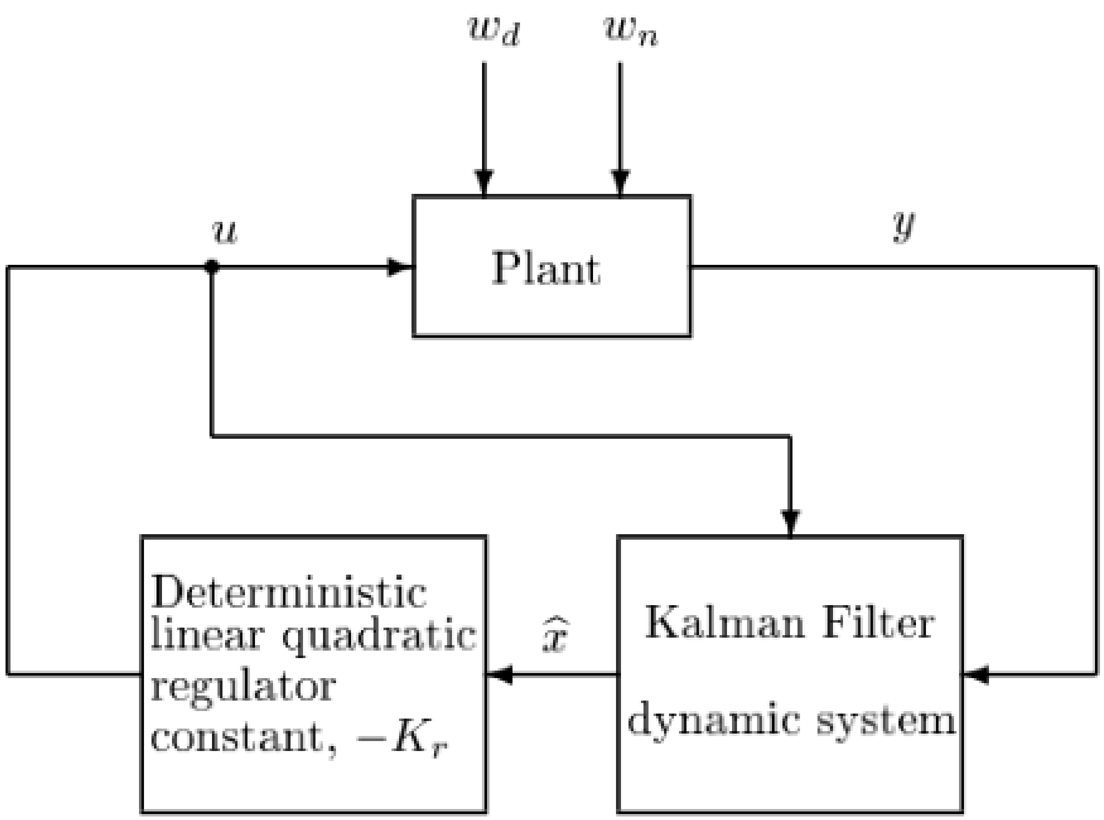

Figure 3: Separation Principle

Controller Design

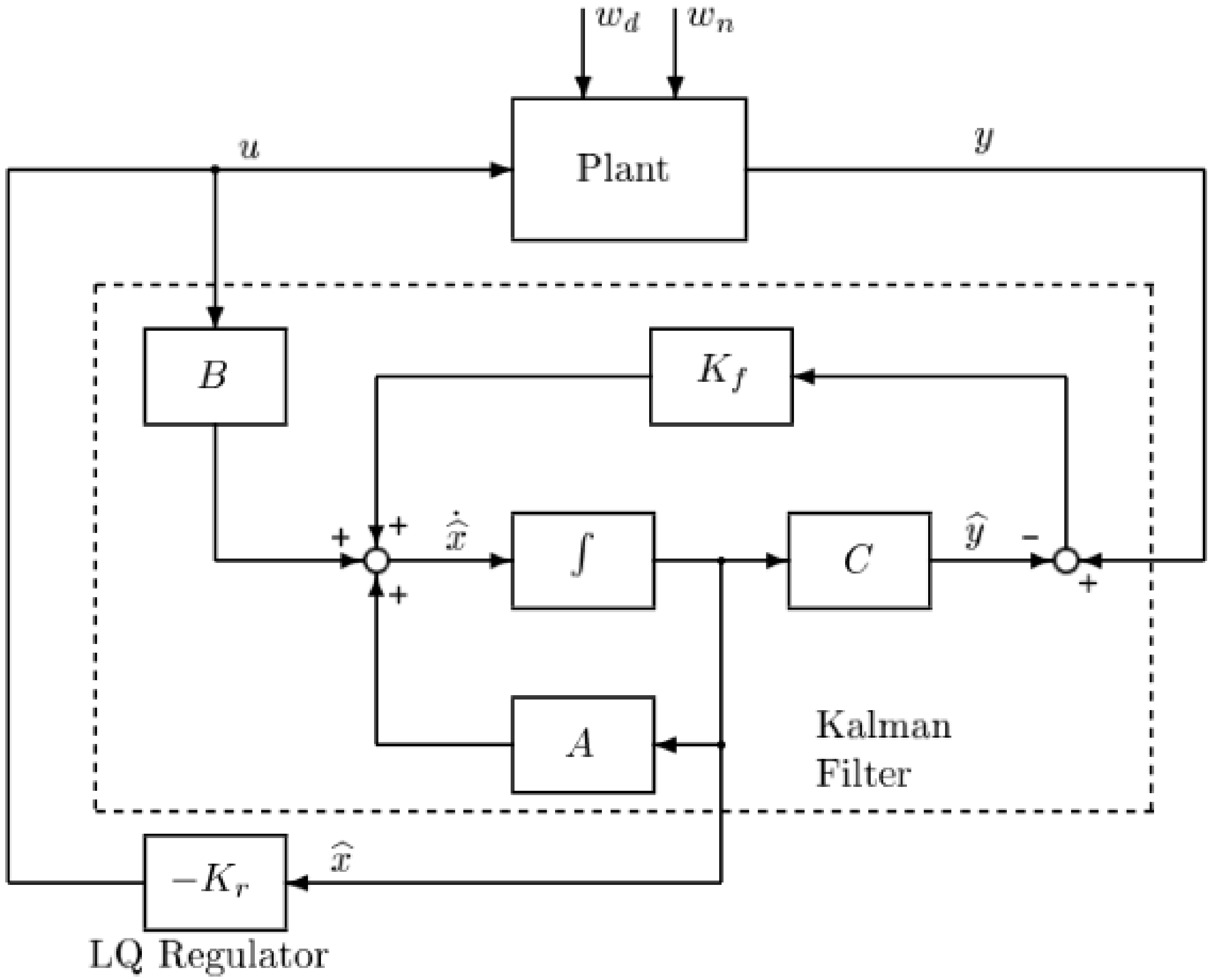

Figure 4: LQG control configuration

Controller Design

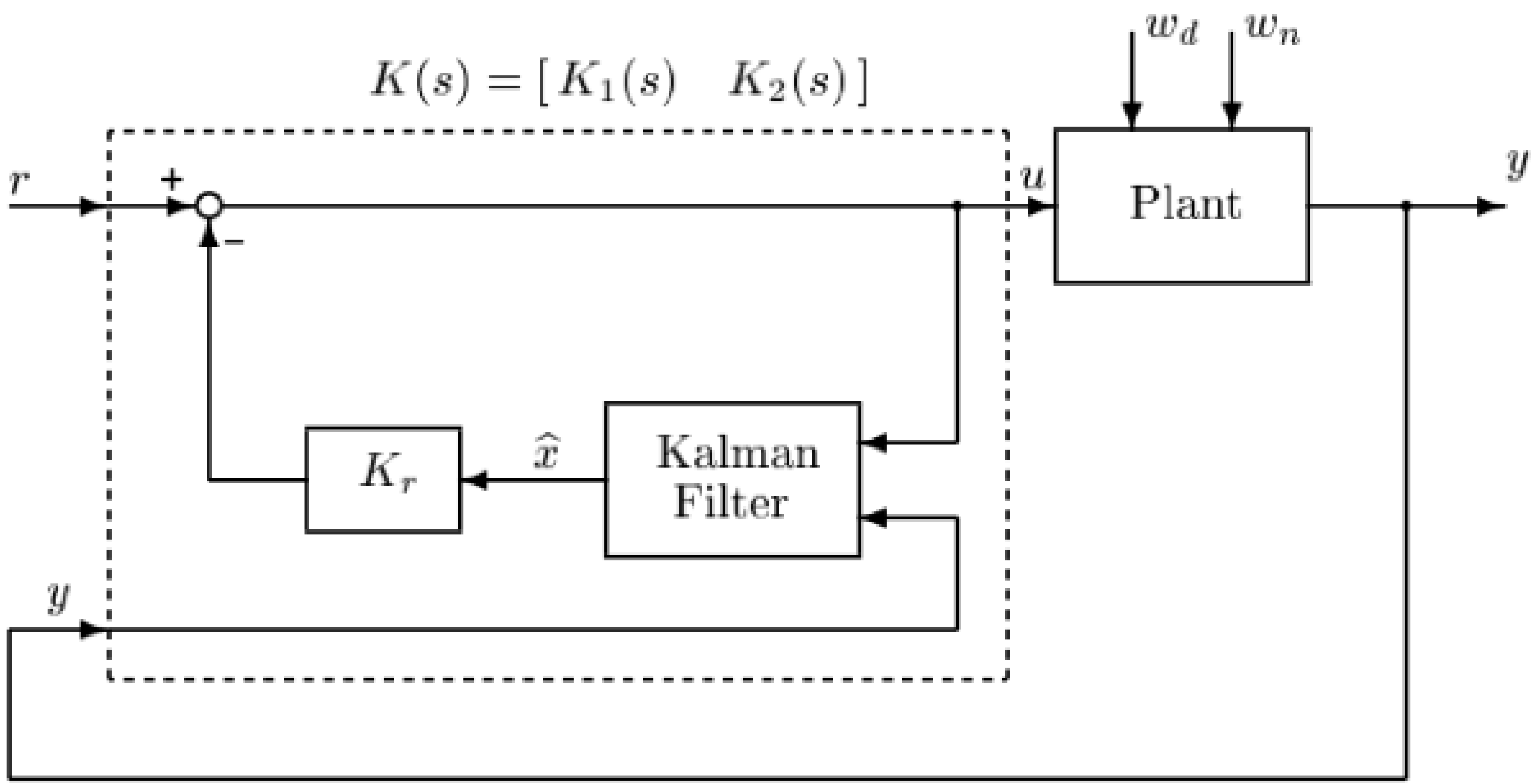

Figure 5: LQG control configuration with reference

Controller Design

7.3 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [9.3]

7.3.1 General control problem formulation [9.3.1]

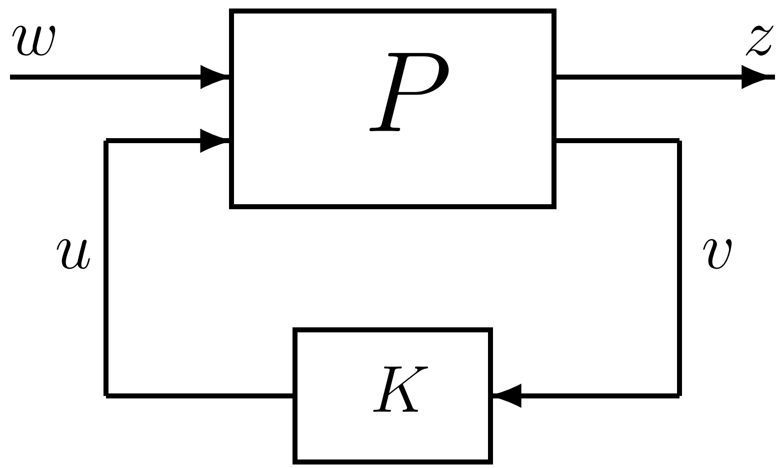

Figure 6: General control configuration

$$\begin{bmatrix} z \\ v \end{bmatrix} = P(s)\begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix} \tag{7.4}$$

$$u = K(s)v \tag{7.5}$$

Controller Design

The state-space realization of the generalized plant P is given by

(7.8)
$$P \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

(7.9)
$$z = F_l(P, K)w$$

where

(7.10)
$$F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$$

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control involve the minimization of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of $F_l(P, K)$ respectively.

Controller Design

7.3.2 $\mathcal{H}_2$ optimal control [9.3.2]

The standard $\mathcal{H}_2$ optimal control problem is to find a stabilizing controller K which minimizes

$$\text{(7.11)} \quad \| F(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(j\omega)^T d\omega}; \qquad F \triangleq F_l(P, K)$$

For a particular problem the generalized plant P will include the plant model, the interconnection structure, and the designer specified weighting functions. This is illustrated for the LQG problem in the next subsection.
Stochastic interpretation: suppose in the general control configuration that the exogenous input w is white noise of unit intensity. That is:

$$\text{(7.12)} \qquad E\left\{w(t)w(\tau)^T\right\} = I\delta(t-\tau)$$

Controller Design

The expected power in the error signal z is then given by:

$$
\begin{aligned}
\text{(7.13)} \quad & E\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} z(t)^T z(t)\,dt\right\} \\
&= \operatorname{tr} E\left\{z(t)z(t)^T\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F(j\omega)^T d\omega
\end{aligned}
$$

(by Parseval's Theorem)

$$
\text{(7.14)} \quad = \|F\|_2^2 = \|F_l(P,K)\|_2^2
$$

Thus, by minimizing the $\mathcal{H}_2$ norm, the output (or error) power of the generalized system, due to a unit intensity white noise input, is minimized; we are minimizing the root-mean-square (rms) value of z.

Controller Design

7.3.3 LQG: a special $\mathcal{H}_2$ optimal controller [9.3.3]

(7.15) $$\dot{x} = Ax + Bu + w_d$$

(7.16) $$y = Cx + w_n$$

where:

(7.17) $$E\left\{\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix}\begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T \end{bmatrix}\right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}\delta(t-\tau)$$

The LQG problem is to find $u = K(s)y$ such that

(7.18) $$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left[x^TQx + u^TRu\right]dt\right\}$$

is minimized with $Q = Q^T \geq 0$ and $R = R^T > 0$.

Controller Design

Define:

$$z = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{7.19}$$

and represent the stochastic inputs w_d, w_n as

$$\begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{\frac{1}{2}} & 0 \\ 0 & V^{\frac{1}{2}} \end{bmatrix} w \tag{7.20}$$

where w is a white noise process of unit intensity. Then the LQG cost function is

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t)^T z(t) dt \right\} = \|F_l(P,K)\|_2^2 \tag{7.21}$$

Controller Design

where

(7.22) $$z(s) = F_l(P, K)w(s)$$

and the generalized plant P is given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc:c} A & W^{\frac{1}{2}} & 0 & B \\ \hline Q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{\frac{1}{2}} \\ \hdashline C & 0 & V^{\frac{1}{2}} & 0 \end{array}\right]$$

Controller Design

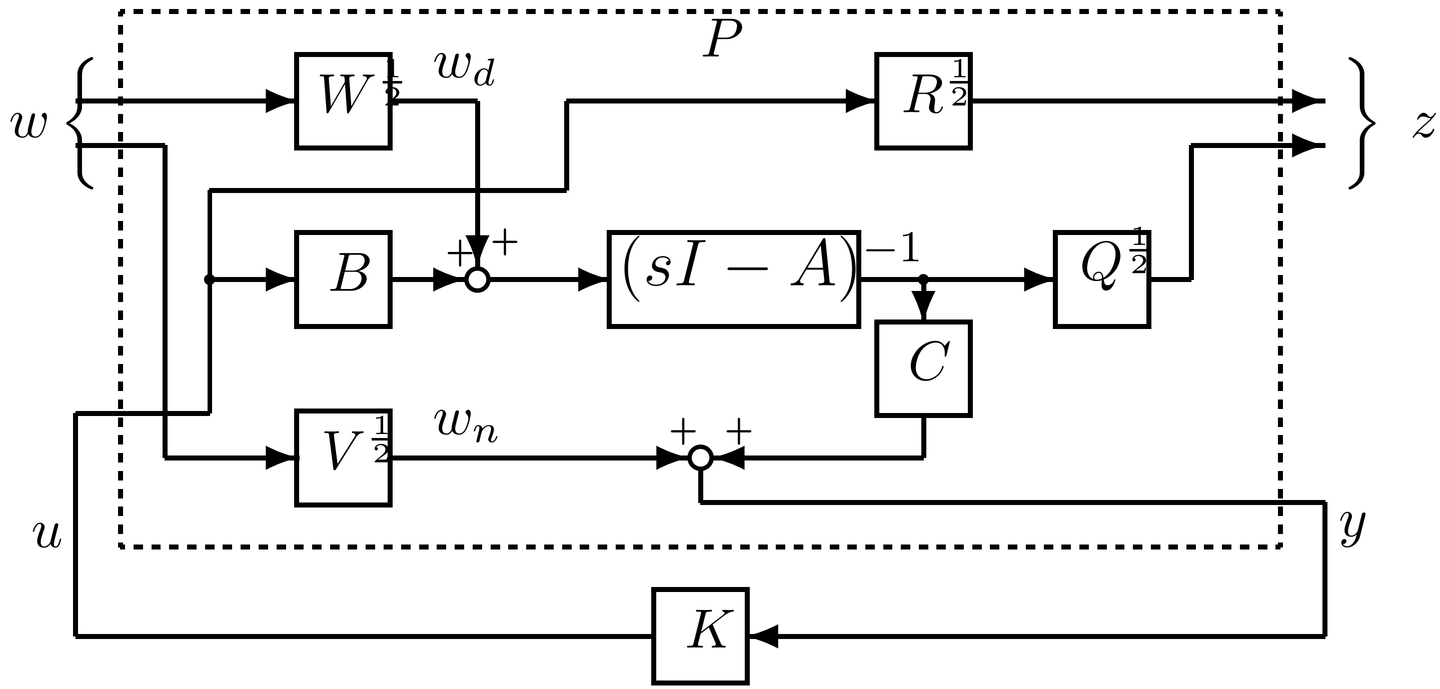

Figure 7: The LQG problem: general control configuration

Controller Design

7.3.4 $\mathcal{H}_\infty$ optimal control [9.3.4]

With reference to the general control configuration of Figure 6, the standard $\mathcal{H}_\infty$ optimal control problem is to find all stabilizing controllers K which minimize

(7.23) $$\|F_l(P,K)\|_\infty = \max_\omega \bar\sigma(F_l(P,K)(j\omega))$$

This has a time domain interpretation as the induced (worst-case) 2-norm. Let $z = F_l(P,K)w$, then

(7.24) $$\|F_l(P,K)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

where $\|z(t)\|_2 = \sqrt{\int_0^\infty \sum_i |z_i(t)|^2 dt}$ is the 2-norm of the vector signal.

Controller Design

It is often computationally (and theoretically) simpler to design a sub-optimal one (i.e. one close to the optimal controller in the sense of the $\mathcal{H}_\infty$ norm). Let $\gamma_{\min}$ be the minimum value of $\|F_l(P,K)\|_\infty$ over all stabilizing controllers K. Then the $\mathcal{H}_\infty$ sub-optimal control problem is: given a $\gamma > \gamma_{\min}$, find all stabilizing controllers K such that

$$\|F_l(P,K)\|_\infty < \gamma$$

Controller Design

7.3.5 Mixed-sensitivity $\mathcal{H}_\infty$ control [9.3.5]

To optimize performance, minimize $\|w_1 S\|_\infty$,
to minimize control inputs, minimize $\|w_2 KS\|_\infty$.
Compromise:

(7.25)
$$\left\| \begin{bmatrix} w_1 S \\ w_2 KS \end{bmatrix} \right\|_\infty$$

General setting: disturbance d as a single exogenous input, error signal $z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$, where $z_1 = W_1 y$ and $z_2 = -W_2 u$, (8).

Controller Design

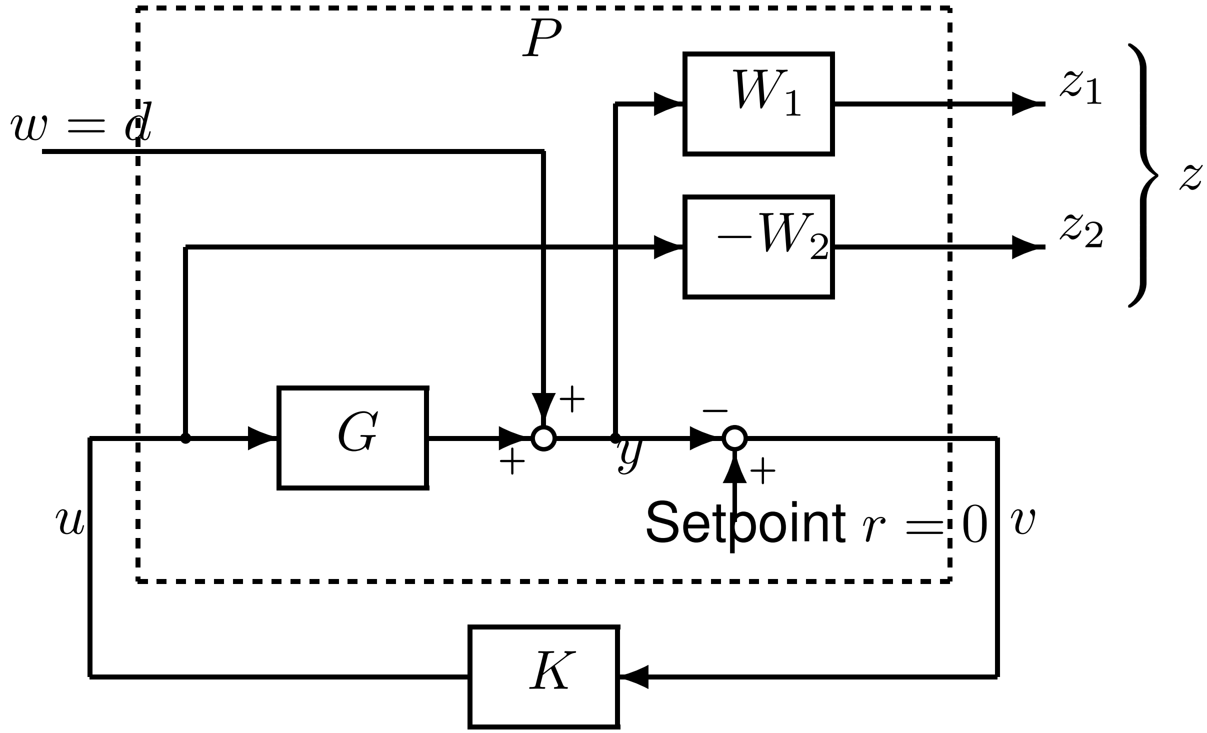

Figure 8: S/KS mixed-sensitivity optimization in standard form (regulation)

Controller Design

Thus $z_1 = W_1 S w$ and $z_2 = W_2 K S w$ and:

(7.26)
$$P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12} = \begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix}$$
$$P_{21} = -I \qquad P_{22} = -G$$

where the partitioning is such that

(7.27)
$$\begin{bmatrix} z_1 \\ z_2 \\ \hdashline v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

and

(7.28)
$$F_l(P, K) = \begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix}$$

Controller Design

Another useful mixed sensitivity optimization problem, is to find a stabilizing controller which minimizes

(7.29)
$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty$$

The S/T mixed-sensitivity minimization problem can be put into the standard control configuration as shown in Figure 9.

(7.30)
$$\begin{array}{ll} P_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} & P_{12} = \begin{bmatrix} -W_1 G \\ W_2 G \end{bmatrix} \\ P_{21} = I & P_{22} = -G \end{array}$$

Controller Design

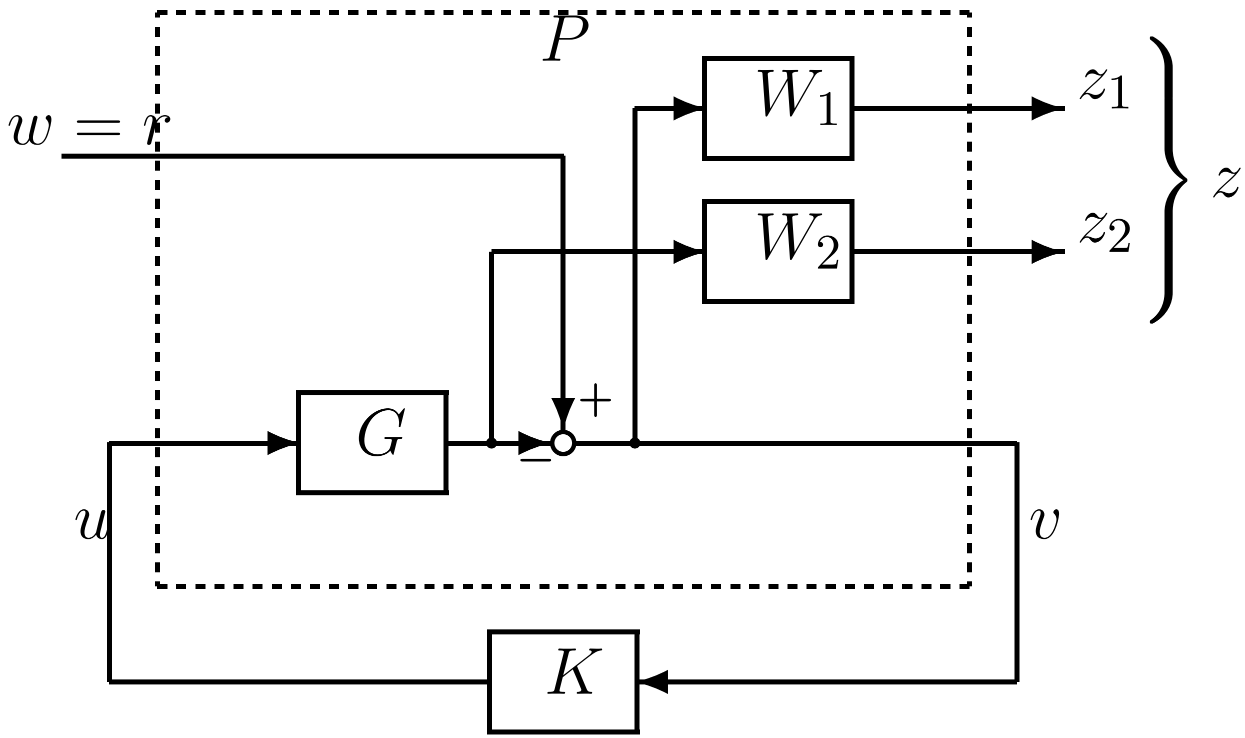

Figure 9: S/T mixed-sensitivity optimization in standard form

Controller Design

7.4 $\mathcal{H}_\infty$ loop-shaping design [9.4]

We need a design procedure more flexible than mixed-sensitivity $\mathcal{H}_\infty$ but not as complicated as μ-synthesis. For simplicity, it should be based on classical loop-shaping ideas.

7.4.1 Coprime Factorization [4.1.5]

A useful way to represent systems is the coprime factorization, which may be used both in state-space and tranfer function forms.

A *right coprime factorization* of G is given by

$$G(s) = N_r(s)M_r^{-1}(s) \tag{7.31}$$

where $N_r(s)$ and M_r are stable coprime transfer functions.

Controller Design

The stability implies that:

- $N_r(s)$ should contain all the RHP-zeros of $G(s)$
- M_r should contain as RHP-zeros all the RHP-poles of $G(s)$

The coprimeness implies that:

- there should be no common RHP-zeros (including the point at infinity) in $N_r(s)$ and M_r, which results in pole-zero cancellation when forming $N_r(s)M_r^{-1}(s)$.

Mathematically, comprimeness means that there exist stable $U_r(s)$ and $V_r(s)$ such that the following Bezout identity is satisfied:

$$U_r N_r + V_r M_r = I \tag{7.32}$$

Controller Design

Similarly, a *left coprime factorization* of G is given by

(7.33) $$G(s) = M_l^{-1}(s)N_l(s)$$

where $N_l(s)$ and M_l are stable coprime transfer functions. That is, there exist stable $U_l(s)$ and $V_l(s)$ such that the following Bezout identity is satisfied:

(7.34) $$N_lU_l + M_lV_l = I$$

Example

(7.35) $$G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

Controller Design

The coprime factorization is NOT unique. We introduce the operator M^* defined as $M^*(s) = M^T(-s)$
Then, $G(s) = N_r(s)M_r^{-1}(s)$ is called a *normalized right coprime factorization* if

$$N_r^* N_r + M_r^* M_r = I \tag{7.36}$$

In this case, $X_r(s) = \begin{bmatrix} M_r \\ N_r \end{bmatrix}$ is a *inner* transfer function which means that $X_r^* X_r = I$.
Then, $G(s) = M_l^{-1}(s)N_l(s)$ is called a *normalized left coprime factorization* if

$$N_l N_l^* + M_l M_l^* = I \tag{7.37}$$

In this case, $X_l(s) = [M_l \quad N_l]$ is a *co-inner* transfer function which means that $X_l X_l^* = I$.

Controller Design

If G has a minimal state-space realization

(7.38) $$G \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Then, a minimal state-space representation of a normalized left coprime factorization is given by

(7.39) $$[N_l(s) M_l(s)] \overset{s}{=} \left[\begin{array}{c|cc} A + HC & B + HD & H \\ \hline R^{-\frac{1}{2}}C & R^{-\frac{1}{2}}D & R^{-\frac{1}{2}} \end{array}\right]$$

where $H = -(BD^T + ZC^T)R^{-1}$, $R = I + DD^T$, and the matrix Z is the unique positive definite solution to the Riccati equation

$$(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0$$

where $S = I + D^TD$.

Controller Design

7.4.2 RS for Coprime Factor Uncertainty [8.6.2]

$$\text{RS} \iff \|M\|_\infty < 1 \tag{7.40}$$

is tight (not conservative) only when there is a single full perturbation block. An “exception" to this is when the uncertainty blocks enter or exit from the same location in the block diagram, because they can be stacked on top of each other or side by side in an overall Δ which is then a full matrix. If we norm-bound the combined (stacked) uncertainty, we then get a tight RS condition in terms of $\|M\|_\infty$.

Controller Design

One important uncertainty that falls into this category is the coprime uncertainty, for which the set of plants is

(7.41) $G_p = (M_l + \Delta_M)^{-1}(N_l + \Delta_N), \quad \|\Delta_N \quad \Delta_M\|_\infty \le \epsilon$

where $G = M_l^{-1} N_l$ is a left coprime factorization of the nominal plant.

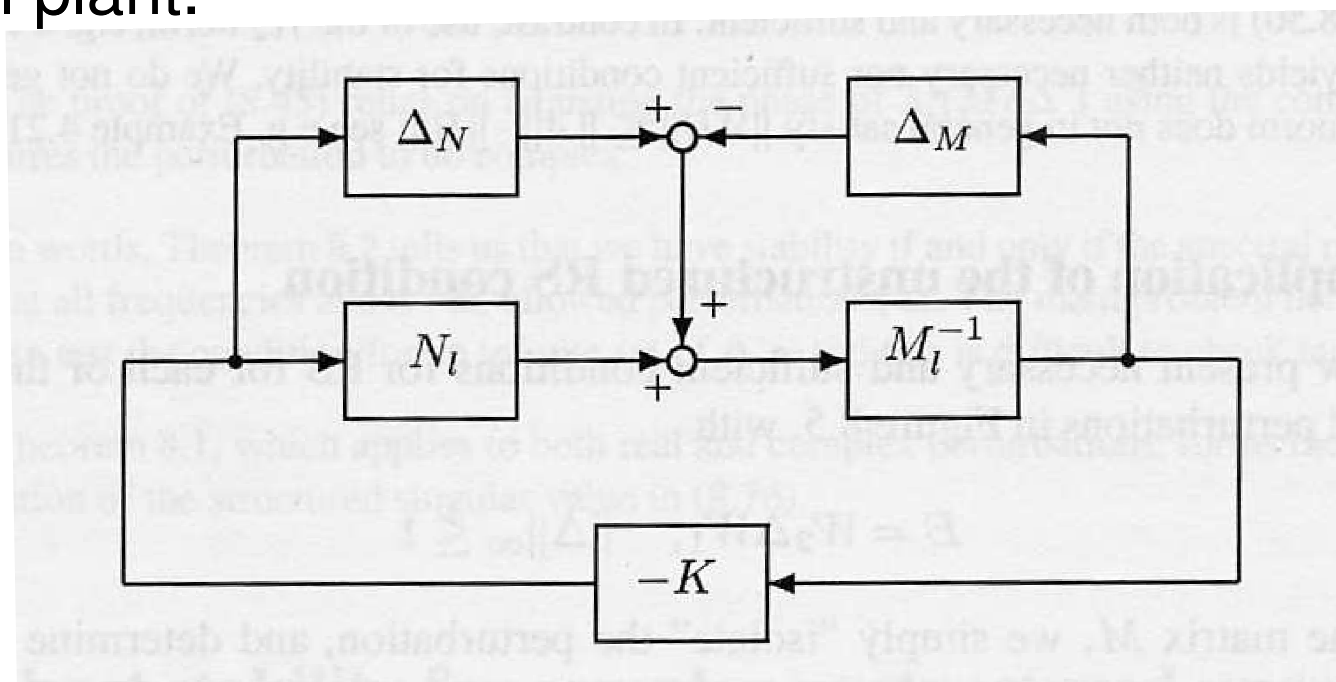

Figure 10: Coprime uncertainty

Controller Design

To test for RS we can rearrange the block diagram into the $M\Delta$ structure with

(7.42) $\Delta = [\Delta_N \quad \Delta_M]; \qquad M = \begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1}$

We then have

(7.43) $$\text{RS } \forall \|\Delta_N \quad \Delta_M\|_\infty \leq \epsilon \iff \|M\|_\infty < 1/\epsilon$$

This result is central to the $\mathcal{H}_\infty$ loop-shaping design procedure.

- Good “generic" uncertainty description when no a-priori uncertainty information is available.
- Often used to maximize the uncertainty magnitude ϵ such that RS is maintained.

Controller Design

7.4.3 Robust Stabilization [9.4.1]

For feedback systems with coprime uncertainty, the stability property is robust *if and only if* the nominal feedback is stable and

$$\gamma_K = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + GK)^{-1} M_l^{-1} \right\|_\infty < 1/\epsilon \quad \forall \| \Delta_N \quad \Delta_M \|_\infty \leq \epsilon \tag{7.44}$$

The lowest achievable γ_K and the corresponding maximum stability margin ϵ were computed analytically

$$\gamma_{min} = \epsilon_{max}^{-1} = \left\{ 1 - \| [N_l \ M_l] \|_H^2 \right\}^{-1/2} = (1 + \rho(XZ))^{1/2} \tag{7.45}$$

where $\| \cdot \|_H$ denotes Hankel norm and ρ denotes the spectral radius (maximum eigenvalue).

Controller Design

For a minimal realization (A, B, C, D) of $G(s)$, Z is the unique positive definite solution to the algebraic Riccati equation

$$(A-BS^{-1}D^TC)Z+Z(A-BS^{-1}D^TC)^T-ZC^TR^{-1}CZ+BS^{-1}B^T = 0$$

(7.46)

where

(7.47) $$R = I + DD^T, \qquad S = I + D^TD$$

and X is the unique positive definite solution to the algebraic Riccati equation

$$(A-BS^{-1}D^TC)^TX+X(A-BS^{-1}D^TC)-XBS^{-1}B^TX+C^TR^{-1}C = 0$$

(7.48)

This formula simplifies considerably for a strictly proper plant, i.e., when D=0;

Controller Design

A controller that guarantees that

(7.49) $$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+GK)^{-1} M_l^{-1} \right\|_\infty \le \gamma$$

for a specified $\gamma > \gamma_{min}$, is given by

$$\begin{aligned} K &\overset{s}{=} \left[\begin{array}{c|c} A + BF + \gamma^2 (L^T)^{-1} Z C^T (C + DF) & \gamma^2 (L^T)^{-1} Z C^T \\ \hline B^T X & -D^T \end{array} \right] \\ F &= -S^{-1}(D^T C + B^T X) \\ L &= (1-\gamma^2) I + XZ \end{aligned}$$

Since we can compute directly γ_{min}, we get an explicit solution by solving just two Riccati equations and avoid the γ-iteration needed to solve the general $\mathcal{H}_\infty$ problem.

Controller Design

7.4.4 A Systematic $\mathcal{H}_\infty$ Loop-Shaping Procedure [9.4.2]

Robust stabilization alone is not much use in practice because the designer is not able to specify any performance requirements. We can add pre- and post-compensators to the plant to shape the open-loop singular values prior to robust stabilization of the “shaped" plant.

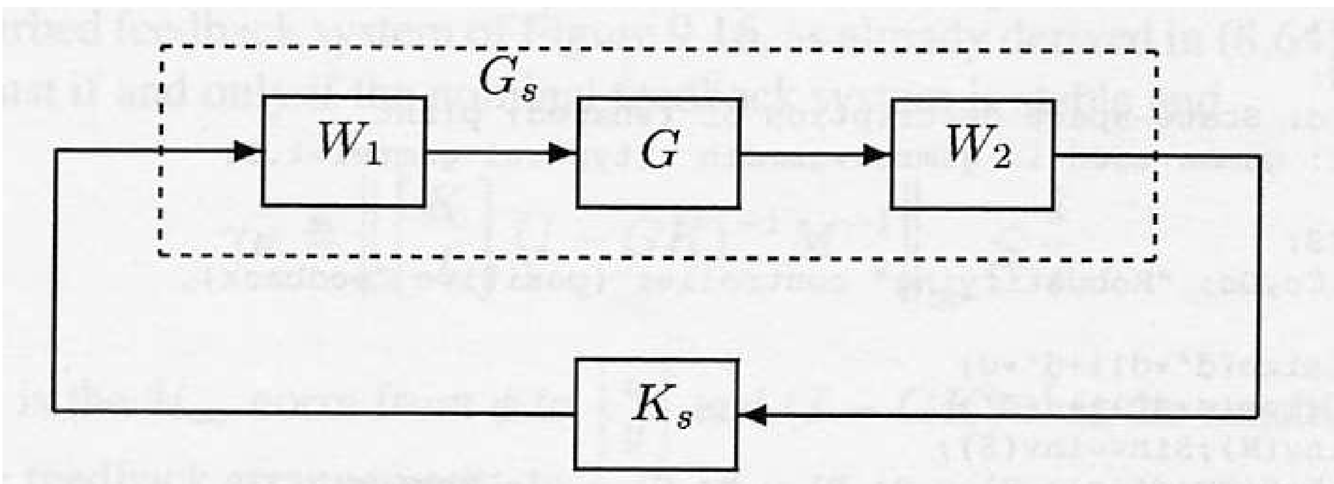

Figure 11: Shaped plant and controller

Controller Design

If W_1 and W_2 are the pre- and post-compensators respectively, the “shaped" plant (initial loop shape) G_s is given by

(7.50) $$G_s = W_2 G W_1$$

The controller K_s is synthesized by solving the robust stabilization problem for the “shaped" plant with a normalized left coprime factorization $G_s = M_s^1 N_s$. The feedback controller for the plant G is then

(7.51) $$K = W_1 K_s W_2$$

This procedure contains all the essential ingredients of classical loop-shaping. The robust stabilization problem can be solved using the formulae presented in the previous section.