FRTN10 Multivariable Control, Lecture 4

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Course Outline

L1-L5 Specifications, models and loop-shaping by hand

1. Introduction and system representations
2. Stability and robustness
3. Specifications and disturbance models
4. Control synthesis in frequency domain
5. Case study

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 4: Control Synthesis in the Frequency Domain

- Review of concepts from lecture 3
  - Calculation of spectral density and variance
  - Spectral factorization
- Control synthesis in frequency domain:
  - Frequency domain specifications
  - Loop shaping
- Feedforward design

[Glad & Ljung] Ch. 6.4–6.6, 8.1–8.2

Example: Spectral density and variance

Assume u to be unit intensity white noise and $G(s) = (s+1)^{-2}$.
What is the spectral density and variance of y?

$$\Phi_u(\omega) = 1$$

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G^*(i\omega) = G(i\omega)G(-i\omega) = \frac{1}{(1+\omega^2)^2}$$

$$\mathbf{E}y^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_y(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(1+\omega^2)^2}d\omega = \frac{1}{4}$$

Example: Spectral density and variance

Alternative (state-space) solution to compute the variance:
$G(s) \Leftrightarrow \texttt{ss}(A, B, C, D)$ with

$$A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad C = \begin{pmatrix} 1/2 & -1/2 \end{pmatrix}, \quad D = 0$$

Lyapunov equation for state covariance $\Pi_x = \mathbf{E}xx^T$:

$$A\Pi_x + \Pi_x A + BB^T = 0 \quad \Rightarrow \quad \Pi = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Variance of y:

$$\mathbf{E}y^2 = \mathbf{E}(Cx)(Cx)^T = C\Pi_x C^T = 1/4$$

Example: Spectral Factorization

Given

$$\Phi_y(\omega) = \frac{1}{\omega^4 + 2\omega^2 + 1}$$

find stable $G(s)$ such that $G(i\omega)G(-i\omega) = \Phi_y(\omega)$

Solution:

$$\frac{1}{\omega^4 + 2\omega^2 + 1} = \frac{1}{(\omega^2 + 1)^2} = \frac{1}{\big((1 + i\omega)(1 - i\omega)\big)^2}$$

$$G(i\omega) = \frac{1}{(1 + i\omega)^2}$$

$$G(s) = \frac{1}{(s + 1)^2}$$

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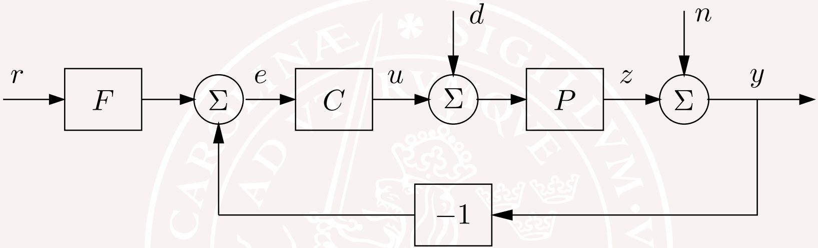
Review: Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Review: Design problem

Find a controller that

A: reduces the effect of load disturbances

B: does not inject too much measurement noise into the system

C: makes the closed loop insensitive to process variations

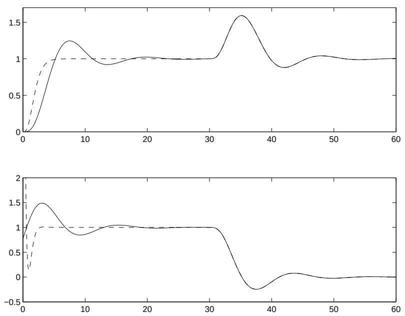
D: makes the output follow the setpoint

It is convenient to use a controller with **two degrees of freedom**, i.e. separate signal transmission from y to u and from r to u. This gives a nice separation of the design problem:

1. First design feedback compensator to deal with A, B, and C.
2. Then design feedforward compensator to deal with D.

Time domain specifications

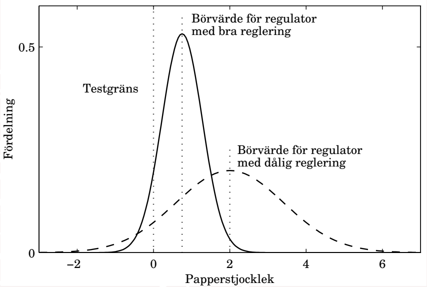
- Specifications on step response (w.r.t reference and/or load disturbance)
  - Rise-time T_r
  - Overshoot M
  - Settling time T_s
  - Static error e_0
  - ...



reference change load disturbance

Stochastic specifications

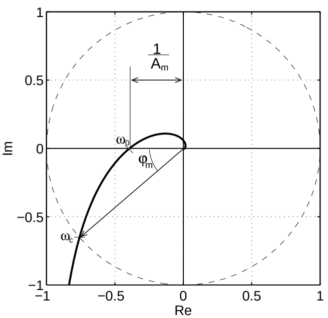
- Output variance
- Control signal variance
- ...



Frequency domain specifications

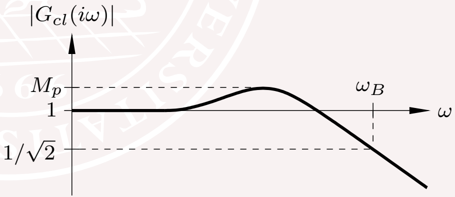
Open-loop specifications

- M_s and M_t circles in Nyquist diagram
- Amplitude margin A_m, phase margin φ_m
- Cross-over frequency ω_c
- ...



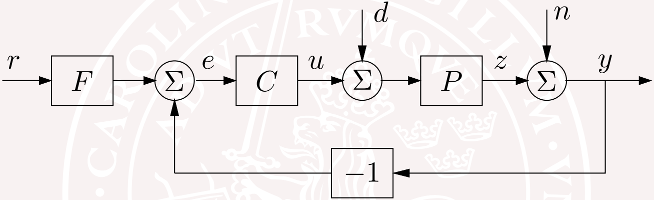
Closed-loop specifications (r to y)

- resonance peak M_p
- bandwidth ω_B
- ...



Frequency domain specifications

Closed-loop specifications, cont'd:



Desired properties:

- Small influence of load disturbance d on z
- Small influence of model errors on z
- Limited amplification of noise n in control u
- Robust stability despite model errors

Frequency domain specifications

Ideally, we would like to design the controller so that

- $$\frac{PCF}{1+PC} = 1$$
- $$\underbrace{\frac{P}{1+PC}}_{=PS} = \underbrace{\frac{1}{1+PC}}_{=S} = \underbrace{\frac{C}{1+PC}}_{=P^{-1}T} = \underbrace{\frac{PC}{1+PC}}_{=T} = 0$$

$S + T = 1$ makes this is impossible to achieve.

Typical compromise:

- Make S small at low frequencies (+ possibly other disturbance dominated frequencies)
- Make T small at high frequencies

Expressing specifications on S and T

Find specifications W_S and W_T for closed-loops transfer functions s.t

$$|S(i\omega)| \le |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \le |W_T^{-1}(i\omega)|$$

(Magnitude corresponds to singular values for MIMO-systems)

Examples:

- $|S(i\omega)| < 1.5$ for $\omega < 5$ Hz
- $|S| < |W_S^{-1}|$ = s/(s+10)
- $|T| < |W_T^{-1}|$ = 10/(s+10)

Expressing specifications on S and T

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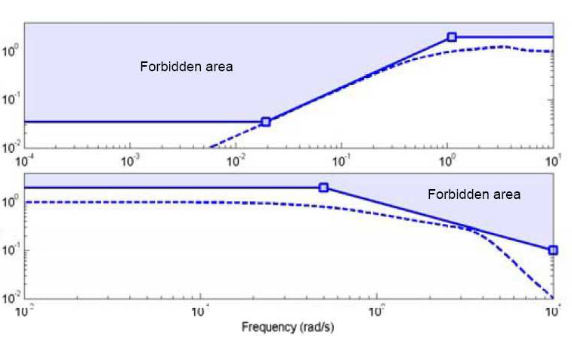


Limitations on specifications

The specifications cannot be chosen independently of each other:

- $S + T = 1$

Fundamental limitations [Lecture 7]:

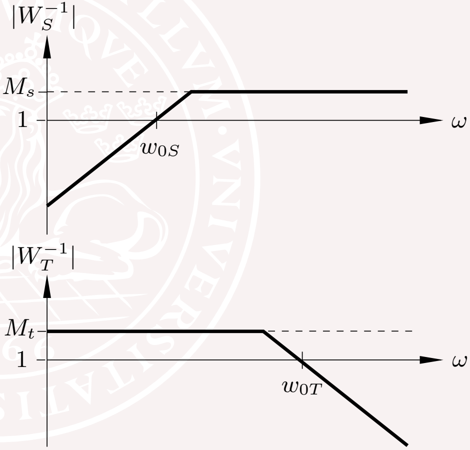
- RHP zero at $z \Rightarrow \omega_{0S} \leq z/2$
- Time delay $T \Rightarrow \omega_{0S} \leq 1/T$
- RHP pole at $p \Rightarrow \omega_{0T} \geq 2p$

Bode's integral theorem:

- The "waterbed effect"

Bode's relation:

- good phase margin requires certain distance between ω_{0S} and ω_{0T}



Loop shaping design

Idea: Look at the **loop-gain** $L = PC$ for design and to translate specifications on S & T into specifications on L

$$S = \frac{1}{1+L} \approx 1/L \qquad \text{if } L \text{ is large}$$

$$T = \frac{L}{1+L} \approx L \qquad \text{if } L \text{ is small}$$

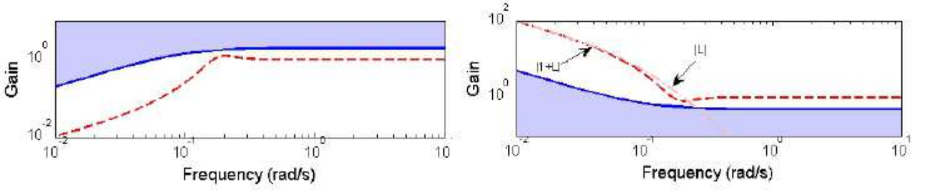
Classical loop shaping:

- design C so that $L = PC$ satisfies constraints on S and T
- how are the specifications related?
- what to do with the regions around cross-over frequency ω_c (where $|L| = 1$)?

Sensitivity vs Loop Gain

$$S = \frac{1}{1+L}$$

$$|S(i\omega)| \le |W_S^{-1}(i\omega)| \iff |1+L(i\omega)| > |W_S(i\omega)|$$



For small frequencies, W_S large $\implies 1+L$ large, and $|L| \approx |1+L|$.

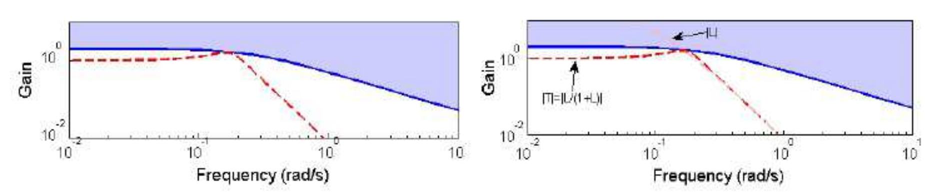
$$|L(i\omega)| \ge |W_S(i\omega)| \quad (approx.)$$

(typically valid for $\omega < \omega_{0S}$)

Complementary Sensitivity vs Loop Gain

$$T = \frac{L}{1+L}$$

$$|T(i\omega)| \le |W_T^{-1}(i\omega)| \iff \frac{|L(i\omega)|}{|1+L(i\omega)|} \le |W_T^{-1}(i\omega)|$$

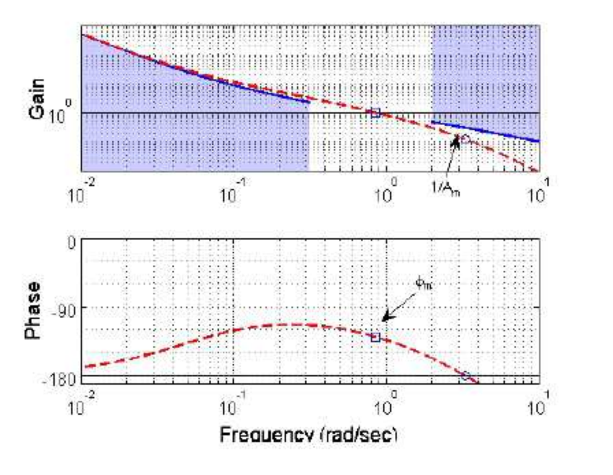


For large frequencies, W_T^{-1} small $\Longrightarrow |T| \approx |L|$

$$|L(i\omega)| \le |W_T^{-1}(i\omega)| \quad (approx.)$$

(typically valid for $\omega > \omega_{0T}$)

Resulting constraints on loop gain L:

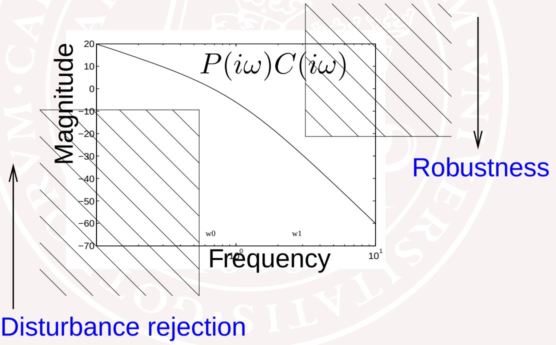


Remark: approximations inexact around cross-over frequency ω_c. In this region, focus is on stability margins A_m, φ_m.

These requirements are to say that the *loop transfer matrix*

$$L = P(i\omega)C(i\omega)$$

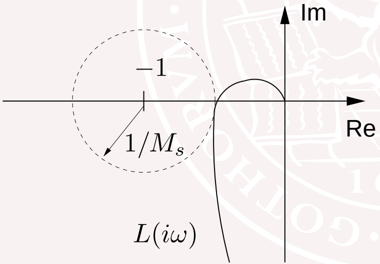
should have large norm $\|P(i\omega)C(i\omega)\|$ at low frequencies and small norm at high frequencies.

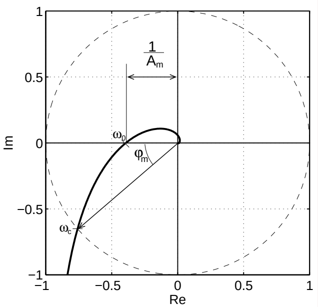


M_s and M_t vs gain and phase margins

Specifying $|S(i\omega)| \le M_s$ and $|T(i\omega)| \le M_t$ gives bounds for the gain and phase margins (but not the other way round!)

$$|S(i\omega)| \le M_s \quad \Longrightarrow \quad A_m > \frac{M_s}{M_s - 1}, \quad \varphi_m > 2 \arcsin \frac{1}{M_s}$$





Q: Why does not A_m and φ_m give bounds on M_s and M_t?

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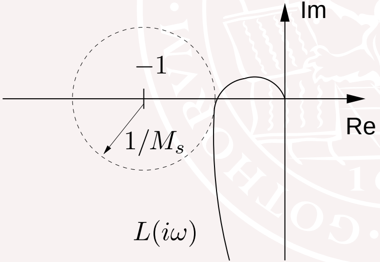
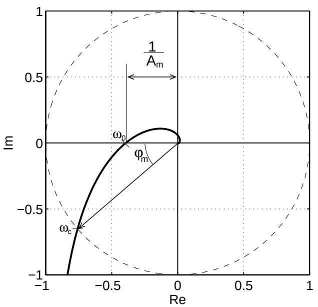




Q: Why does not A_m and φ_m give bounds on M_s and M_t?

Classical loop shaping

Map specifications on requirements on loop gain L.

- Low-frequency specifications from W_S
- High-frequency specifications from W_T^{-1}
- Around cross-over frequency, mapping is crude
  - Position cross-over frequency (constrained by W_S, W_T)
  - Adjust phase margin (e.g. from M_s, M_t specifications)

Lead-lag compensation

Shape loop gain $L = PC$ using a compensator C composed of

- Lag (phase retarding) elements

$$C_{lag} = \frac{s+a}{s+a/M}, \quad M > 1$$

- Lead (phase advancing) elements

$$C_{lead} = N\frac{s+b}{s+bN}, \quad N > 1$$

- Gain

$$K$$

Typically

$$C = K\frac{s+a}{s+a/M} \cdot N\frac{s+b}{s+bN}$$

Properties of leads-lag elements

- Lag (phase retarding) elements
  - Reduces static error
  - Reduces stability margin
- Lead (phase advancing) elements
  - Increased speed by increased ω_c
  - Increased phase

    $\Longrightarrow$ May improve stability
- Gain
  - Translates magnitude curve
  - Does not change phase curve

See "Collection of Formulae" for lead-lag link diagrams

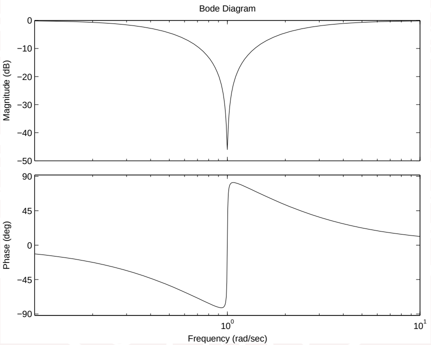
Iterative lead-lag design

- Step 1: Lag (phase retarding) element
  - Add phase retarding element to get low-frequency asymptote right
- Step 2: Phase advancing element
  - Use phase advancing element to obtain correct phase margin
- Step 3: Adjust gain
  - Usually need to "lift up" or "push down" amplitude curve to obtain the desired cross-over frequency.

Adjusting the gain in Step 3 leaves the phase unaffected, but may ruin low-frequency asymptote (need to revise lag element) $\Longrightarrow$ An iterative method!

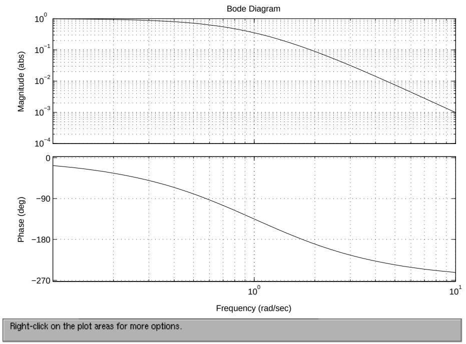
Example of other compensation link:

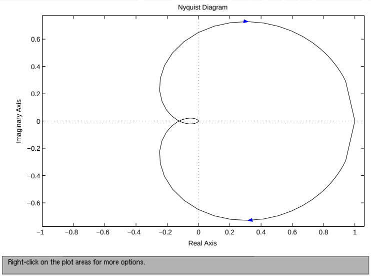
Notch filter $\dfrac{s^2 + 0.01s + 1}{s^2 + 2s + 1}$

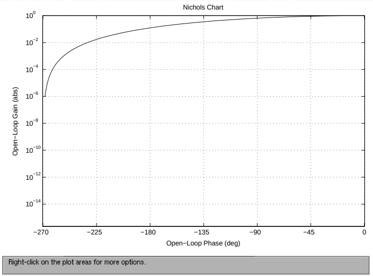


Bode, Nyquist and Nichols diagrams

$$\log |PC| = \log |P| + \log |C|$$
$$\arg\{PC\} = \arg\{P\} + \arg\{C\}$$







Lecture 4: Control Synthesis in the Frequency Domain

- Review of concepts from lecture 3
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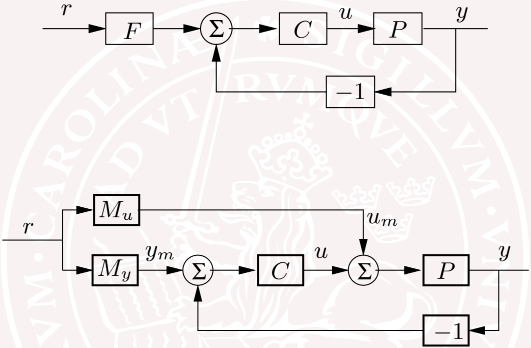
Feedforward design



The reference signal r specifies the desired value of y.

Ideally

$$\frac{P(s)C(s)}{1+P(s)C(s)}F(s) \approx 1$$

Equivalently

$$F(s) \approx \frac{1 + P(s)C(s)}{P(s)C(s)}$$

Exact equality is generally impossible because of pole excess in P.

The simplest and most common approximation is to use a constant gain

$$F = \frac{1 + P(0)C(0)}{P(0)C(0)}$$

A more advanced option is

$$F(s) = \frac{1 + P(s)C(s)}{P(s)C(s)(sT+1)^d}$$

for some suitable time constant T and d large enough to make F proper and implementable.

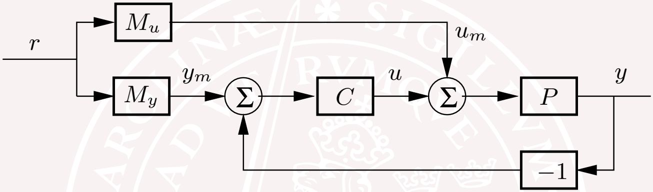
Example

$$P(s) = \frac{1}{(s+1)^4} \qquad F(s) = \frac{1+P(s)C(s)}{P(s)C(s)(sT+1)^d}$$

The closed loop transfer function from r to u then becomes

$$\frac{C(s)}{1+P(s)C(s)} F(s) = \frac{(s+1)^4}{(sT+1)^4}$$

which has low-fq gain 1, but gain $1/T^4$ for $\omega \longrightarrow \infty$.



Notice that M_u and M_y can be viewed as generators of the desired output y_m and the inputs u_m which corresponds to y_m.

Design of Feedforward revisited

The transfer function from r to $e = y_m - y$ is $(M_y - PM_u)S$

Ideally, M_u should satisfy $M_u = M_y/P$. This condition does not depend on C!

Since $M_u = M_y/P$ should be stable, causal and not include derivatives we find that

- Unstable process zeros must be zeros of M_y
- Time delays of the process must be time delays of M_y
- The pole excess of M_y must not be smaller than the pole excess of P

Take process limitations into account!

Example of Feedforward Design revisited

If

$$P(s) = \frac{1}{(s+1)^4} \qquad M_y(s) = \frac{1}{(sT+1)^4}$$

then

$$M_u(s) = \frac{M_y(s)}{P(s)} = \frac{(s+1)^4}{(sT+1)^4} \qquad \frac{M_u(\infty)}{M_u(0)} = \frac{1}{T^4}$$

Fast response (T small) requires high gain of M_u.

Bounds on the control signal limit how fast response we can obtain.

Summary

Frequency domain design:

- Good mapping between S, T and $L = PC$ at low and high frequencies (mapping around cross-over frequency less clear)
- Simple relation between C and $L \Longrightarrow$ easy to shape L!
- Lead-lag control: iterative adjustment procedure
- What if closed-loop specifications are not satisfied?
  - we made a poor design (did not iterate enough), or
  - the specifications are not feasible (fundamental limitations in Lecture 7)
- Later in the course::
  - Use optimization to find stabilizing controller that satisfies constraints, if such a controller exists

Feedforward design

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