Lecture 9: Linear Quadratic Control

- Dynamic Programming
- Riccati equation
- Optimal State Feedback
- Stability and Robustness

The sections 9.1-9.4 + 5.7 in the book treat essentially the same material as we cover in lecture 9-11. However, the main derivation of the LQG controller in appendix 9A is different.

Course outline

L1-L5 Purpose, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

- L9 **LQ - optimal state feedback**
- L10 Kalman filter - optimal state observer
  LQG = LQ-design + Kalman filter
- L11 more on LQG - output feedback control

L12-L14 Controller optimization: Numerical approach

Math Repetition

Suppose the matrix Q is symmetric: $Q = Q^T$. Then

- $Q > 0$ means that $x^T Q x > 0$ for any $x \neq 0$
  - True iff all eigenvalues of Q are positive.
  - We say that Q is positive definite.

- $Q \geq 0$ means that $x^T Q x \geq 0$ for any $x \neq 0$
  - True iff all eigenvalues of Q are non-negative.
  - We say that Q is positive semidefinite.

Math Repetition

The trace of a matrix is the sum of all diagonal elements:

$$\text{trace } Q = \sum_{i}^{n} Q_{ii}$$

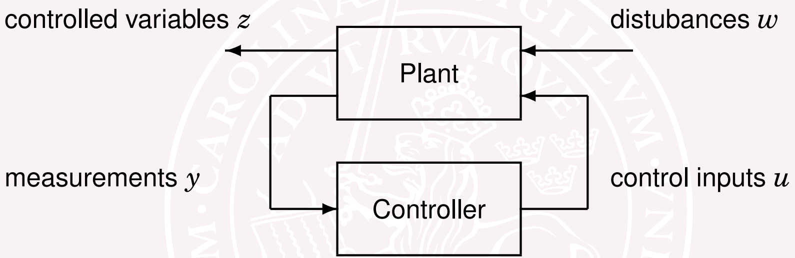
A useful property of the matrix trace:

$$\text{trace } ABC = \text{ trace } CAB = \text{ trace } BCA$$

Parseval's formula: Suppose that $f(t)$ and $g(t)$ have finite energy and that their Laplacerespectively. Then

$$2\pi \int_{-\infty}^{\infty} f(t)^* g(t) dt = \int_{-\infty}^{\infty} F(i\omega)^* G(i\omega) d\omega$$

A General Optimization Setup

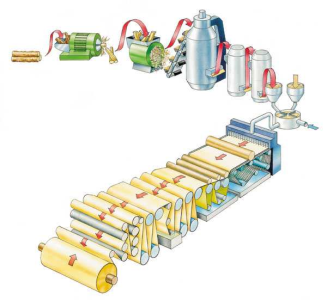


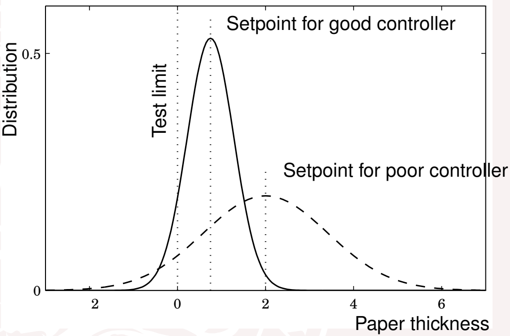
The objective is to find a controller that optimizes the transfer matrix $G_{zw}(s)$ from disturbances w to controlled outputs z.

Lecture 9-11: Problems with analytic solutions

Lectures 12-14: Problems with numeric solutions

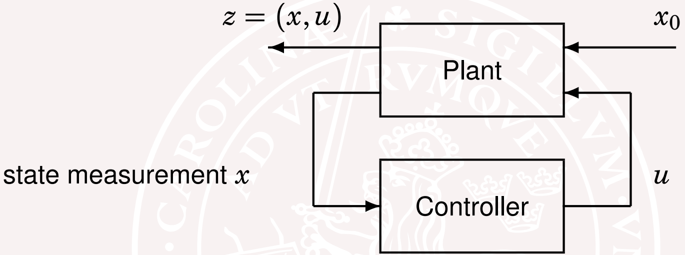
Thickness control in paper machine





All paper production below the test limit is wasted.
Good control allows for lower setpoint with the same waste. The average thickness is lower, which saves significant costs.

Today's problem: State Feedback



Minimize $$\int_0^\infty \Big(x(t)^T Q_1 x(t) + 2x(t)^T Q_{12} u(t) + u(t)^T Q_2 u(t) \Big) dt$$

subject to $$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

Mini-problem

Determine u_0 and u_1 as functions of x_0 if the objective is to minimize

$$x_1^2 + x_2^2 + u_0^2 + u_1^2$$

when

$$\begin{aligned} x_1 &= x_0 + u_0 \\ x_2 &= x_1 + u_1 \end{aligned}$$

Hint: Go backwards in time.

Solution to Mini-problem

$$f(u_0, u_1) = x_1^2 + x_2^2 + u_0^2 + u_1^2 = (\underbrace{x_0 + u_0}_{x_1})^2 + ((\underbrace{x_0 + u_0)}_{x_1} + u_1)^2 + u_0^2 + u_1^2$$

$$= 2{x_0}^2 + (2u_1 + 4u_0)\,x_0 + 2u_0u_1 + 2{u_1}^2 + 3{u_0}^2$$

$$\frac{\partial f}{\partial u_0} = 4x_0 + 2u_1 + 6u_0 = 0$$

$$\frac{\partial f}{\partial u_1} = 2x_0 + 2u_0 + 4u_1 = 0$$

(Don't forget to check whether maximum or minimum...)

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} -4x_0 \\ -2x_0 \end{bmatrix} \Longrightarrow \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}x_0 \\ -\frac{1}{5}x_0 \end{bmatrix} \Longrightarrow f_{min} = \frac{3}{5}x_0$$

Note: This sequence depends on the initial value only (no feedback).

For robustness it is prefererable to find a feedback solution!

Quadratic Optimal Cost

The optimal cost on the time interval $[T_1, \infty]$ is quadratic:

$$x^T S x = \min_{\mathbf{u}} \int_{T_1}^{\infty} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} dt$$

$$\text{when} \begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{x}(T_1) = x \end{cases}$$

Dynamic programming, Richard E. Bellman 1957

An optimal trajectory on the time interval $[T_1, T]$ must be optimal also on each of the subintervals $[T_1, T_1 + \epsilon]$ and $[T_1 + \epsilon, T]$.

Dynamic programming in linear quadratic control

$$\mathbf{x}(T_1) = x, \qquad \mathbf{x}(T_1+\epsilon) = x + (Ax+Bu)\epsilon$$

$$\begin{aligned}
x^TSx &= \min_{\mathbf{u}} \int_{T_1}^{\infty} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} dt \\
&= \min_{\mathbf{u}} \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \epsilon + \int_{T_1+\epsilon}^{\infty} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} dt \right\} \\
&= \min_{\mathbf{u}} \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \epsilon + \Big[x + (Ax+Bu)\epsilon\Big]^T S \Big[x + (Ax+Bu)\epsilon\Big] \right\}
\end{aligned}$$

by definition of S. Neglecting ϵ^2 gives **Bellman's equation**:

$$0 = \min_u \Big[\Big(x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u \Big) + 2x^TS(Ax+Bu) \Big]$$

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Completion of squares

The scalar case: Suppose $c > 0$.

$$ax^2 + 2bxu + cu^2 = x\left(a - \frac{b^2}{c}\right)x + \left(u + \frac{b}{c}x\right)c\left(u + \frac{b}{c}x\right)$$

is minimized by $u = -\frac{b}{c}x$. The minimum is $\left(a - b^2/c\right)x^2$.

The matrix case: Suppose $Q_u > 0$. Then

$$\begin{aligned}
&x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u \\
&= (u + Q_u^{-1} Q_{xu}^T x)^T Q_u (u + Q_u^{-1} Q_{xu}^T x) + x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x
\end{aligned}$$

is minimized by $u = -Q_u^{-1} Q_{xu}^T x$. The minimum is
$x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$.

The Riccati Equation

Completion of squares in Bellman's equation gives

$$\begin{aligned}
0 &= \min_u \Big(\big(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \big) + 2x^T S(Ax + Bu) \Big) \\
&= \min_u \Big(x^T [Q_1 + A^T S + SA] x + 2x^T [Q_{12} + SB] u + u^T Q_2 u \Big) \\
&= x^T \Big(Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \Big) x
\end{aligned}$$

with minimum attained for $u = -Q_2^{-1}(SB + Q_{12})^T x$.

The equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

is called the *algebraic Riccati equation*

Jocopo Francesco Riccati, 1676–1754

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Linear Quadratic Optimal Control

Problem:

Minimize $$\int_0^\infty \Big(x(t)^T Q_1 x(t) + 2x(t)^T Q_{12} u(t) + u(t)^T Q_2 u(t)\Big) dt$$

subject to $$\dot{x} = Ax(t) + Bu(t), \qquad x(0) = x_0$$

Solution: Assume (A, B) controllable. Then there is a unique $S > 0$ solving the Riccati equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with $L = Q_2^{-1}(SB + Q_{12})^T$.
The minimal value is $x_0^T S x_0$.

Remark: The feedback gain L does not depend on x_0

Example: First order system

For $\dot{x}(t) = u(t)$, $x(0) = x_0$,

Minimize	$\int_0^\infty \left\{ x(t)^2 + \rho u(t)^2 \right\} dt$
Riccati equation	$0 = 1 - S^2/\rho \quad \Rightarrow \quad S = \sqrt{\rho}$
Controller	$L = S/\rho = 1/\sqrt{\rho} \quad \Rightarrow \quad u = -x/\sqrt{\rho}$
Closed loop system	$\dot{x} = -x/\sqrt{\rho} \quad \Rightarrow \quad x = x_0 e^{-t/\sqrt{\rho}}$
Optimal cost	$\int_0^\infty \left\{ x^2 + \rho u^2 \right\} dt = x_0^T S x_0 = x_0^2 \sqrt{\rho}$

What values of ρ give the fastest response? Why?
What values of ρ give smallest optimal cost? Why?

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Theorem: Stability of the closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

is positive definite and that there exists a positive-definite steady-state solution S to the algebraic Riccati equation. Then the optimal controller $u(t) = -Lx(t)$ gives an asymptotically stable closed-loop system $\dot{x}(t) = (A - BL)x(t)$.

Proof:

$$\begin{aligned} \frac{d}{dt} x(t)^T S x(t) &= 2x^T S \dot{x} = 2x^T S(Ax + Bu) \\ &= -\Big(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \Big) < 0 \text{ for } x(t) \neq 0 \end{aligned}$$

Hence $x(t)^T S x(t)$ is decreasing and tends to zero as $t \to \infty$.

How to solve the LQ problem in Matlab

```
[L,S,E] = LQR(A,B,Q,R,N) calculates the optimal gain
matrix L such that the state-feedback law u = -Lx
minimizes the cost function

    J = Integral x'Qx + u'Ru + 2*x'Nu dt

subject to the system dynamics dx/dt = Ax + Bu

    E = EIG(A-B*L)


LQRD solves the corresponding discrete time problem
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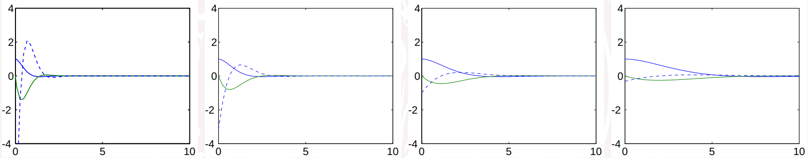
Example – Double integrator

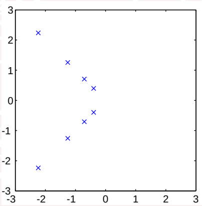
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_2 = \rho \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

States and inputs (dotted) for $\rho = 0.01$, $\rho = 0.1$, $\rho = 1$, $\rho = 10$

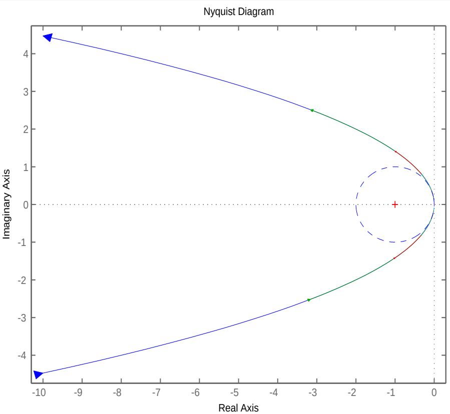


Closed loop poles:

$$s = 2^{-1/2}\rho^{-1/4}(-1 \pm i)$$



Stability robustness of optimal state feedback



Notice that the distance from $L(i\omega I - A)^{-1}B$ to -1 is never smaller than 1. This is always true(!) for linear quadratic optimal state feedback when $Q_1 > 0$, $Q_{12} = 0$ and $Q_2 = \rho > 0$ is scalar. Hence the phase margin is at least $60°$.

Proof of stability robustness

Using the Riccati equation

$$0 = Q_1 + A^T S + SA - L^T Q_2 L \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

it is straightforward to verify that

$$\left[I + L(i\omega - A)^{-1}B\right]^* Q_2 \left[I + L(i\omega - A)^{-1}B\right] = \begin{bmatrix}(i\omega - A)^{-1}B \\ I\end{bmatrix}^* \begin{bmatrix}Q_1 & Q_{12} \\ Q_{12}^* & Q_2\end{bmatrix} \begin{bmatrix}(i\omega - A)^{-1}B \\ I\end{bmatrix}$$

In particular, with $Q_1 > 0$, $Q_{12} = 0$, $Q_2 = \rho > 0$

$$\begin{aligned}\left[1 + L(i\omega - A)^{-1}B\right]^* \rho \left[1 + L(i\omega - A)^{-1}B\right] &= B^T[(i\omega - A)^{-1}]^* Q_1 (i\omega - A)^{-1}B + \rho \\ &\geq \rho\end{aligned}$$

Dividing by ρ gives

$$|1 + L(i\omega - A)^{-1}B| \geq 1$$

Lecture 9: Summary

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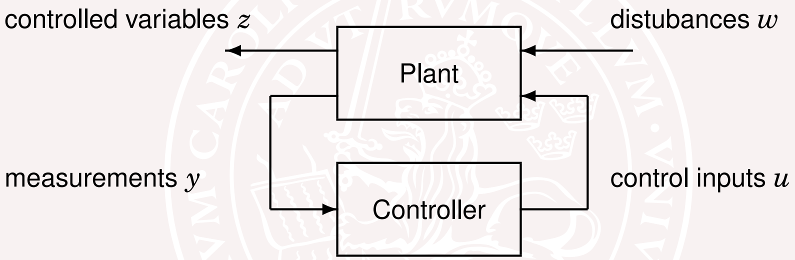
Next Lecture: Linear Quadratic Gaussian Control



For a linear plant, minimize a quadratic function of the map from disturbance w to controlled variable z