1. Proponha um algoritmo computacional que compute automaticamente, com uma precisão preestabelecida, o logaritmo neperiano de x segundo a série de potências:

$$\ln(x)=\sum_{k=1}^{\infty}(-1)^{k-1}\cdot\frac{(x-1)^k}{k}=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots$$

```
log_neperiano(x, δ) :=  m ← 0
                        y ← x
                        while |y| > 2      if  x ≥ 2
                            m ← m + 1
                            y ← √y
                        y ← y − 1
                        S ← y
                        T ← y
                        i ← 1
                        crit ← y
                        while  crit > δ
                            T ← −(i·y)/(i + 1)·T
                            S ← S + T
                            crit ← |T/S|   if  |S| < 1
                            crit ← |T|   otherwise
                            i ← i + 1
                        S ← 2^m·S  if  m > 0
                        R_0 ← S
                        R_1 ← i
                        R
```

$x := 20$ $\qquad\qquad \delta := 10^{-8}$

$R := \text{log\_neperiano}(x,\delta)$ $\qquad R = \begin{pmatrix} 2.99573 \\ 21 \end{pmatrix}$ $\qquad R_0 - \ln(x) = 7.30829\times 10^{-9}$

2-)Proponha um algoritmo computacional que compute automaticamente, com uma precisão preestabelecida, o logaritmo neperiano de x segundo a série de potências:

$$\ln(x) = 2\cdot\sum_{k=1}^{\infty}\frac{1}{2\cdot k-1}\cdot\left(\frac{x-1}{x+1}\right)^{2\cdot k-1} = 2\left[\left(\frac{x-1}{x+1}\right)+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^{3}+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^{5}+\frac{1}{7}\left(\frac{x-1}{x+1}\right)^{7}+\cdots\right]$$

```
log_neperiano2(x, δ) := | y ← (x − 1)/(x + 1)
                        | S ← y
                        | T ← y
                        | y ← y^2
                        | i ← 1
                        | crit ← y
                        | crit ← .9·δ  if  |y| < 10^−12
                        | while  crit > δ
                        |   | T ← ((2·i − 1)/(2·i + 1))·y·T
                        |   | S ← S + T
                        |   | crit ← |T/S|   if  |S| < 1
                        |   | crit ← |T|   otherwise
                        |   | i ← i + 1
                        | R_0 ← S + S
                        | R_1 ← i
                        | R
```

$x := 20$ $\qquad\qquad \delta := 10^{-8}$

$R := \text{log\_neperiano}_2(x,\delta)$ $\qquad R = \begin{pmatrix} 2.99573 \\ 69 \end{pmatrix}$ $\qquad R_0 - \ln(x) = -6.79321\times 10^{-8}$

$\ln(x) = 2.99573$

3-)Proponha um algoritmo computacional que compute automaticamente, com uma precisão preestabelecida, a potência q de x [ x número real e positivo e q número real qualquer] segundo a série de potências:

$$x^q = 1 + q\cdot(x-1) + \frac{q\cdot(q-1)}{2!}\cdot(x-1)^2 + \frac{q\cdot(q-1)\cdot(q-2)}{3!}\cdot(x-1)^3 + \frac{q\cdot(q-1)\cdot(q-2)\cdot(q-3)}{4!}\cdot(x-1)^4 \cdots$$

esta série é convergente para : $0 < x \le 2$ .

```
Potencia(x, q, δ) := | y ← x − 1  if x ≤ 1
                     | y ← 1/x − 1  otherwise
                     | flag ← 0
                     | q ← 1 + q
                     | i ← 1
                     | T ← 1
                     | S ← 1
                     | crit ← 1
                     | crit ← .9·δ  if |y| < 10^(−12)
                     | while crit > δ
                     |   | T ← ((q − i)/i)·y·T
                     |   | S ← S + T
                     |   | crit ← |T/S|  if |S| < 1
                     |   | crit ← |T|  otherwise
                     |   | flag ← 1  if crit < δ
                     |   | i ← i + 1
                     | R_1 ← i
                     | R_0 ← S  if x ≤ 1
                     | R_0 ← 1/S  otherwise
                     | R
```

$x := 7 \qquad q := \frac{1}{3} \qquad \delta := 10^{-14} \qquad \sqrt[3]{x} = 1.91293$

$R := \text{Potencia}(x, q, \delta) \qquad R = \begin{pmatrix} 1.91293 \\ 162 \end{pmatrix} \qquad R_0 - \sqrt[3]{x} = -9.63674 \times 10^{-14}$

Questões 4 a 7 da lista de exercícios do Capítulo 2 $TOL := 10^{-12}$

4) Encontre a aproximação de Padé de quarta ordem ($m = n = 4$) para a função $f(x) = \cos(x)$ e calcule o máximo erro absoluto no intervalo [-1, 1].

Solução: u = $x^2$ equivale a $m = n = 2$

$m := 2$ $n := m$ $N := m + n$ $N = 4$

$f(x) := \cos(x)$

$$fx(x) := f(x) \text{ series}, 2N + 1 \;\rightarrow 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

$$fs(u) := fx(\sqrt{u}) \rightarrow \frac{u^4}{40320} - \frac{u^3}{720} + \frac{u^2}{24} - \frac{u}{2} + 1$$

$a := 0$ $b := 0$ $a_0 := 1$

$i := 1..N$ $b_i := 0$ $a_i := 0$

$$c := fs(u) \text{ coeffs} \;\rightarrow \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{24} \\ -\frac{1}{720} \\ \frac{1}{40320} \end{pmatrix}$$

Montando o sistema para calcular os coeficientes a e b:

$i := 1..n$ $j := 1..n$

$B_{i-1,j-1} := c_{i+j-1}$

$d_{i-1} := -c_{i+n}$

$bb := B^{-1} \cdot d$ $b_0 := c_0$ $b_i := bb_{n-i}$

$$b = \begin{pmatrix} 1 \\ \frac{11}{252} \\ \frac{13}{15120} \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0.044 \\ 8.598 \times 10^{-4} \\ 0 \\ 0 \end{pmatrix}$$

```
A(b) := | for i ∈ 1..n
        |   for j ∈ 0..i
        |     p[i-1,j] ← b[i-j]
        | p
```

$i := 0..n$

$d_i := c_i$

$aa := A(b) \cdot d$ $a_j := aa_{j-1}$

$$a = \begin{pmatrix} 1 \\ -\frac{115}{252} \\ \frac{313}{15120} \\ 0 \\ 0 \end{pmatrix} \qquad a = \begin{pmatrix} 1 \\ -0.456 \\ 0.021 \\ 0 \\ 0 \end{pmatrix}$$

Verificando:

$k := 0..N$

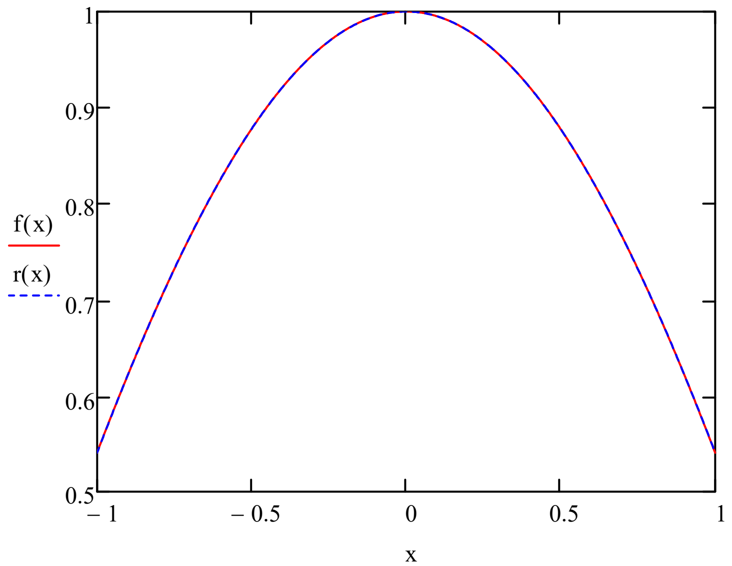
$$\sum_{i=0}^{k}\left(c_i \cdot b_{k-i}\right) - a_k =$$

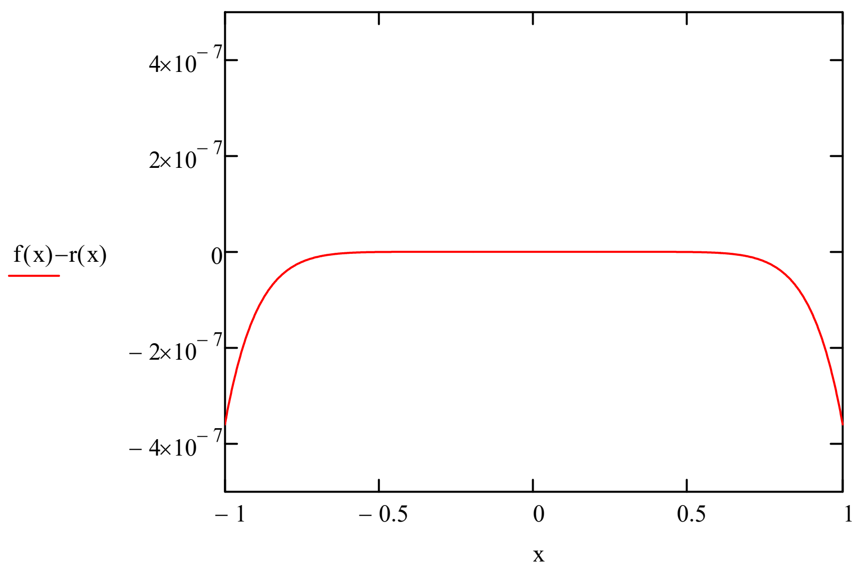
| |
|---|
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |

$$ru(u) := \frac{\sum_{i=0}^{n}\left(a_i \cdot u^i\right)}{\sum_{i=0}^{m}\left(b_i \cdot u^i\right)} \text{ collect } \rightarrow \frac{313\cdot u^2 - 6900\cdot u + 15120}{13\cdot u^2 + 660\cdot u + 15120}$$

$$r(x) := ru\left(x^2\right) \rightarrow \frac{313\cdot x^4 - 6900\cdot x^2 + 15120}{13\cdot x^4 + 660\cdot x^2 + 15120}$$

$$erro\_max := \left|f(1) - r(1)\right| = 3.599 \times 10^{-7}$$



5) Encontre a aproximação de Padé com $m = 4$ e $n = 5$ para a função $f(x) = \text{sen}(x)$ e calcule o máximo erro absoluto no intervalo [-1, 1].

Solução: $g(x) = f(x)/x$ e $u = x^2$ equivale a $m = n = 2$

$m := 2$ $\quad n := m$ $\quad N := m + n$ $\quad N = 4$

$f(x) := \sin(x)$

$$fx(x) := f(x) \text{ series}, 2N+1 \;\rightarrow x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}$$

$$fs(u) := \frac{fx(\sqrt{u})}{\sqrt{u}} \text{ simplify } \rightarrow \frac{u^4}{362880} - \frac{u^3}{5040} + \frac{u^2}{120} - \frac{u}{6} + 1$$

$a := 0$ $\quad b := 0$ $\quad a_0 := 1$

$i := 1..N$ $\quad b_i := 0$ $\quad a_i := 0$

$$c := fs(u) \text{ coeffs } \rightarrow \begin{pmatrix} 1 \\ -\frac{1}{6} \\ \frac{1}{120} \\ -\frac{1}{5040} \\ \frac{1}{362880} \end{pmatrix}$$

Montando o sistema para calcular os coeficientes a e b:

$i := 1..n$ $\quad j := 1..n$ $\quad B := 0$ $\quad d := 0$

$B_{i-1,j-1} := c_{i+j-1}$

$d_{i-1} := -c_{i+n}$

$bb := B^{-1} \cdot d$ $\quad b_0 := c_0$ $\quad b_i := bb_{n-i}$

$$b = \begin{pmatrix} 1 \\ \frac{13}{396} \\ \frac{5}{11088} \\ 0 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0.033 \\ 4.509 \times 10^{-4} \\ 0 \\ 0 \end{pmatrix}$$

```
A(b) := | for i ∈ 1..n
        |   for j ∈ 0..i
        |     p[i-1,j] ← b[i-j]
        | p
```

$i := 0..n$

$d_i := c_i$

$aa := A(b) \cdot d$ $\quad a_j := aa_{j-1}$

$$a = \begin{pmatrix} 1 \\ -\frac{53}{396} \\ \frac{551}{166320} \\ 0 \\ 0 \end{pmatrix} \quad a = \begin{pmatrix} 1 \\ -0.134 \\ 3.313 \times 10^{-3} \\ 0 \\ 0 \end{pmatrix}$$

Verificando:

$k := 0 .. N$

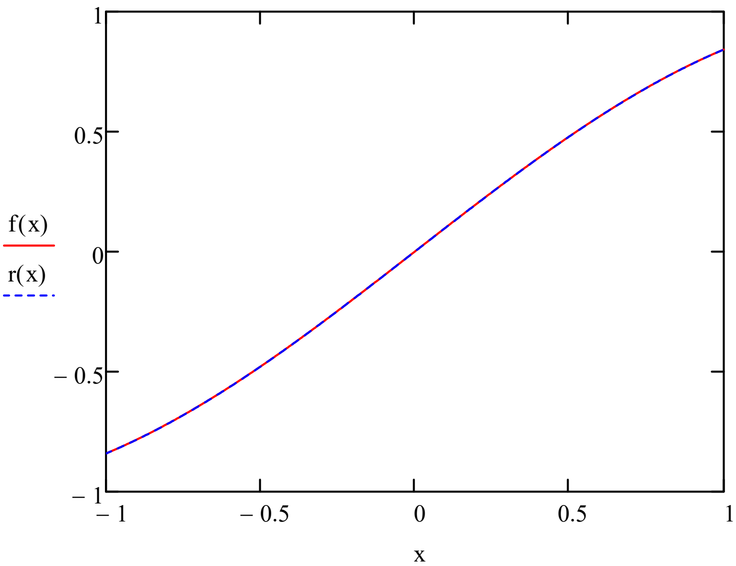
$$\sum_{i=0}^{k} \left(c_i \cdot b_{k-i}\right) - a_k =$$

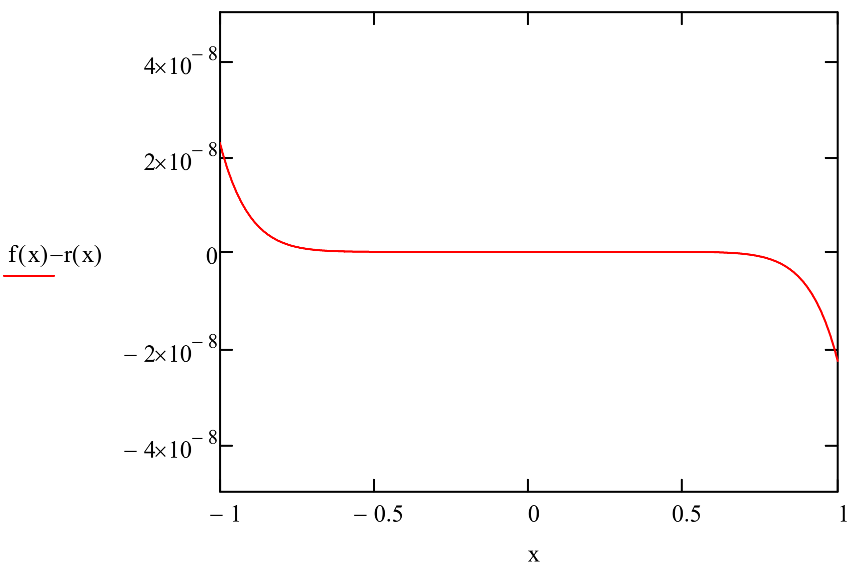
| |
|---|
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |

$$ru(u) := \frac{\sum_{i=0}^{n} \left(a_i \cdot u^i\right)}{\sum_{i=0}^{m} \left(b_i \cdot u^i\right)} \text{ collect } \rightarrow \frac{551 \cdot u^2 - 22260 \cdot u + 166320}{75 \cdot u^2 + 5460 \cdot u + 166320}$$

$$r(x) := x\,ru\left(x^2\right) \rightarrow \frac{x \cdot \left(551 \cdot x^4 - 22260 \cdot x^2 + 166320\right)}{75 \cdot x^4 + 5460 \cdot x^2 + 166320}$$

$$erro\_max := \left|f(1) - r(1)\right| = 2.273 \times 10^{-8}$$



6) Encontre a aproximação de Padé com $m = 2$ e $n = 3$ para a função $f(x) = x\, ln(x)$ e calcule o máximo erro absoluto no intervalo [1/2, 2].

Solução: $g(x) = f(x)/(x-1)$ e u = $(x-1)^2$ equivale a $m = n = 2$

$m := 2 \qquad n := m \qquad N := m + n \qquad N = 4$

$f(x) := x\cdot \ln(x)$

$$fx(x) := f(x) \text{ series}, x = 1, N+1 \;\rightarrow -1 + x + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{20}$$

$$fs(u) := \frac{fx(u+1)}{u} \text{ simplify } \rightarrow \frac{u^3}{12} - \frac{u^4}{20} - \frac{u^2}{6} + \frac{u}{2} + 1$$

$a := 0 \qquad b := 0 \qquad a_0 := 1$

$i := 1 .. N \qquad b_i := 0 \qquad a_i := 0$

$$c := fs(u) \text{ coeffs } \rightarrow \begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{12} \\ -\frac{1}{20} \end{pmatrix}$$

Montando o sistema para calcular os coeficientes a e b:

$i := 1 .. n \qquad j := 1 .. n \qquad B := 0 \qquad d := 0$

$B_{i-1,j-1} := c_{i+j-1}$

$d_{i-1} := -c_{i+n}$

$bb := B^{-1}\cdot d \qquad b_0 := c_0 \qquad b_i := bb_{n-i}$

$$b = \begin{pmatrix} 1 \\ \frac{4}{5} \\ \frac{1}{10} \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0.8 \\ 0.1 \\ 0 \\ 0 \end{pmatrix}$$

```
A(b) := | for i ∈ 1 .. n
        |   for j ∈ 0 .. i
        |     p[i-1,j] ← b[i-j]
        | p
```

$i := 0 .. n$

$d_i := c_i$

$aa := A(b)\cdot d \qquad a_j := aa_{j-1}$

$$a = \begin{pmatrix} 1 \\ \frac{13}{10} \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \qquad a = \begin{pmatrix} 1 \\ 1.3 \\ 0.333 \\ 0 \\ 0 \end{pmatrix}$$

Verificando:

$k := 0..N$

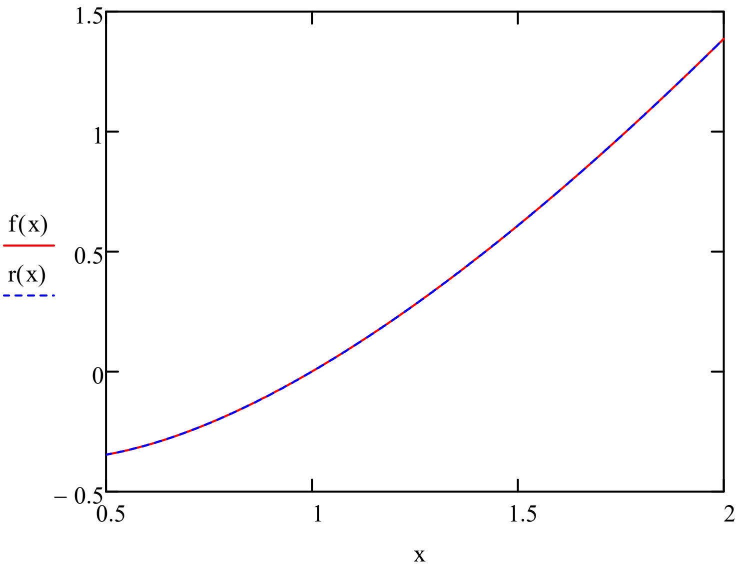
$$\sum_{i=0}^{k} \left(c_i \cdot b_{k-i}\right) - a_k =$$

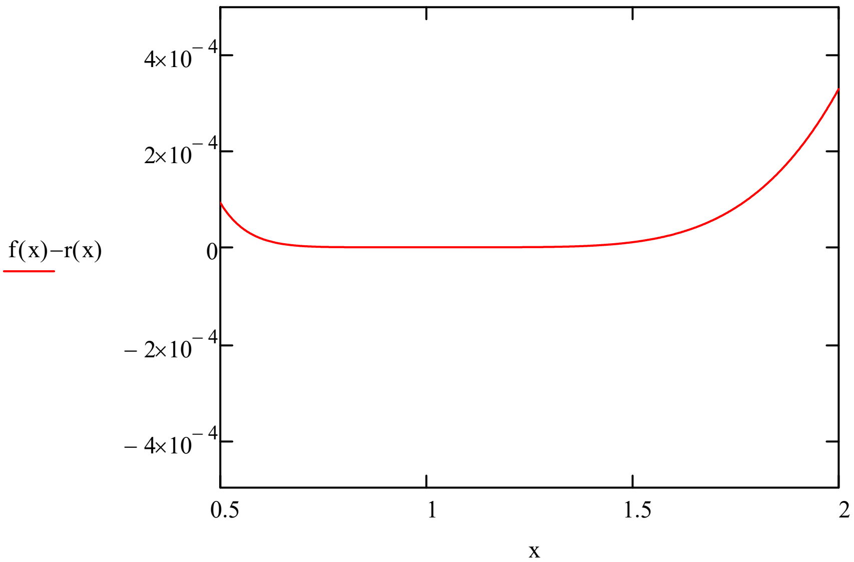
| |
|---|
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |

$$ru(u) := \frac{\sum_{i=0}^{n} \left(a_i \cdot u^i\right)}{\sum_{i=0}^{m} \left(b_i \cdot u^i\right)} \text{ collect } \rightarrow \frac{10 \cdot u^2 + 39 \cdot u + 30}{3 \cdot u^2 + 24 \cdot u + 30}$$

$$r(x) := (x-1) \cdot ru(x-1) \text{ collect}, x-1 \rightarrow \frac{10 \cdot (x-1)^3 + (x-1) \cdot (39 \cdot x - 9)}{24 \cdot x + 3 \cdot (x-1)^2 + 6}$$

$$erro\_max := \left| f(2) - r(2) \right| = 3.294 \times 10^{-4}$$



7) Encontre a aproximação de Padé de terceira ordem ($m = n = 3$) para a função $f(x) = \exp(-x)$ e calcule o máximo erro absoluto no intervalo [-1, 1].
Solução:

$m := 3 \qquad n := m \qquad N := m + n \qquad N = 6$

$f(x) := \exp(-x)$

$fx(x) := f(x) \text{ series}, N + 1 \rightarrow 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{72}$

$a := 0 \qquad b := 0 \qquad a_0 := 1$

$i := 1 .. N \qquad b_i := 0 \qquad a_i := 0$

$$c := fx(u) \text{ coeffs} \rightarrow \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{24} \\ -\frac{1}{120} \\ \frac{1}{720} \end{pmatrix}$$

Montando o sistema para calcular os coeficientes a e b:

$i := 1 .. n \qquad j := 1 .. n \qquad B := 0 \qquad d := 0$

$B_{i-1,j-1} := c_{i+j-1}$

$d_{i-1} := -c_{i+n}$

$bb := B^{-1} \cdot d \qquad b_0 := c_0 \qquad b_i := bb_{n-i}$

$$b = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{10} \\ \frac{1}{120} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0.5 \\ 0.1 \\ 8.333 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A(b) := \left| \begin{array}{l} \text{for } i \in 1 .. n \\ \quad \text{for } j \in 0 .. i \\ \quad\quad p_{i-1,j} \leftarrow b_{i-j} \\ p \end{array} \right.$$

$i := 0 .. n$

$d_i := c_i$

$aa := A(b) \cdot d \qquad a_j := aa_{j-1}$

$$a = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{10} \\ -\frac{1}{120} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad a = \begin{pmatrix} 1 \\ -0.5 \\ 0.1 \\ -8.333 \times 10^{-3} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Verificando:

$k := 0 .. N$

$$\sum_{i=0}^{k} \left(c_i \cdot b_{k-i}\right) - a_k =$$

| |
|---|
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |
| 0 |

$$r(x) := \frac{\sum_{i=0}^{n} \left(a_i \cdot x^i\right)}{\sum_{i=0}^{m} \left(b_i \cdot x^i\right)} \text{ collect } \rightarrow -\frac{x^3 - 12\cdot x^2 + 60\cdot x - 120}{x^3 + 12\cdot x^2 + 60\cdot x + 120}$$

$$erro\_max := \left|f(-1) - r(-1)\right| = 2.803 \times 10^{-5}$$